S©MR

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

*Vol. 23, No. 2, pp. 1063–1077 (2026)*
https://doi.org/10.33048/semi.2026.23.064

УДК 512.542
MSC 20D10, 20D25

# THE INTERSECTION OF $\mathfrak{F}$-MAXIMAL SUBGROUPS OF A FINITE GROUP AND ITS COMPUTATION

**V. I. MURASHKA** AND **Y. A. KUPTSOVA**

*Communicated by* Yu.B. Gorhskov

**Abstract:** We investigate the properties of the intersection $\mathrm{Int}_{\mathfrak{F}}(G)$ of all $\mathfrak{F}$-maximal subgroups of a finite group $G$ for a hereditary formation $\mathfrak{F}$ of finite groups with the help of the theory of $\mathfrak{F}$-subnormal subgroups. For a wide family of formations $\mathfrak{F}$ we obtain the $\mathfrak{F}$-hypercenter like characterization of $\mathrm{Int}_{\mathfrak{F}}(G)$. As corollaries we obtain the results of A. N. Skiba (2011), J. C. Beidleman and H. Heineken (2011) about $\mathrm{Int}_{\mathfrak{F}}(G)$ for a hereditary saturated formation $\mathfrak{F}$. We discuss the computational aspects of $\mathrm{Int}_{\mathfrak{F}}(G)$.

**Keywords:** finite group, formation, $\mathfrak{F}$-subnormal subgroup, weak $\mathfrak{F}$-subnormalizer, intersection of $\mathfrak{F}$-maximal subgroups, $\mathfrak{F}$-hypercenter.

## 1 Introduction and the main results

All groups considered here are finite. The concept of the hypercenter as the final term of the upper central series plays an important role in the modern group theory. The development of this concept within the framework of the theory of formations was carried out in the works of R. Baer [1] for the class of all supersolvable groups, B. Huppert [2] for local formations, L. A. Shemetkov for graduated formations [3]

Murashka, V.I., Kuptsova, Y.A. The intersection of $\mathfrak{F}$-maximal subgroups of a finite group and its computation.
© 2026 Murashka V.I., Kuptsova Y.A.
The work is supported by Belarusian Republican Foundation for Fundamental Research (BRFFR-RSF M, project F23RNFM-63).
*Received July, 16, 2026, Published November, 23, 2026.*

and appeared in its final form in the monograph of L. A. Shemetkov and A. N. Skiba [4]. Let $\mathfrak{F}$ be a formation. Recall (see [4] or [5, p. 6]) that a chief factor $H/K$ of a group $G$ is called $\mathfrak{F}$-central provided that the semidirect product of $H/K$ with $G/C_G(H/K)$ belongs to $\mathfrak{F}$, where $G/C_G(H/K)$ acts on $H/K$ by conjugation. The largest normal subgroup of $G$ such that all chief factors of $G$ below it are $\mathfrak{F}$-central in $G$ is called the $\mathfrak{F}$-hypercenter (see [4, p.142] or [5]) and is denoted by $\mathrm{Z}_{\mathfrak{F}}(G)$.

Before the mentioned works R. Baer in 1966 [6] proposed a different approach to define the $\mathfrak{e}$-hypercenter for a not necessary finite group and a group-theoretical property $\mathfrak{e}$. The $\mathfrak{e}$-hypercenter $\mathfrak{h}_{\mathfrak{e}}G$ of $G$ is the set of elements $g$ of $G$ such that $\langle g, E\rangle$ is a subgroup of some $\mathfrak{e}$-subgroup of $G$ for any $\mathfrak{e}$-subgroup $E$ of $G$ [6, §4]. From [7] it follows that if $\mathfrak{e}$ is nilpotency, then $\mathfrak{h}_{\mathfrak{e}}G$ is the hypercenter of $G$. Note that if $\mathfrak{e}$ is the property of a group to belong to a class $\mathfrak{F}$, then $\mathfrak{h}_{\mathfrak{e}}G$ is the intersection $\mathrm{Int}_{\mathfrak{F}}(G)$ of all $\mathfrak{F}$-maximal subgroups of $G$. Various properties of this subgroup were studied by A. N. Skiba [8, 9], W. Guo and A. N. Skiba [10], J. C. Beidleman and H. Heineken [11], A. Lucchini and D. Nemmi [12], V. I. Murashka and A. F. Vasil'ev [13]. One of the key properties of the $\mathfrak{F}$-hypercenter, in particular the hypercenter, is: $\mathrm{Z}_{\mathfrak{F}}(G/\mathrm{Z}_{\mathfrak{F}}(G)) \simeq 1$ for any group $G$. An analogue of this property for $\mathrm{Int}_{\mathfrak{F}}(G)$ was obtained in [8, 9] when $\mathfrak{F}$ is a hereditary saturated formation.

**Theorem 1.** *There exists a hereditary solubly saturated formation $\mathfrak{F}$ and a group $G$ such that $\mathrm{Int}_{\mathfrak{F}}(G/\mathrm{Int}_{\mathfrak{F}}(G)) \not\simeq 1$.*

Therefore the problem to describe all hereditary formations $\mathfrak{F}$ such that $\mathrm{Int}_{\mathfrak{F}}(G/\mathrm{Int}_{\mathfrak{F}}(G)) \simeq 1$ holds for any group $G$ seems natural. Our solution to this problem is closely related to the concept of an $\mathfrak{F}$-subnormal subgroup. Recall (see [14, Definition 6.1.2]) that a subgroup $H$ of $G$ is called $\mathfrak{F}$-subnormal in $G$ (shortly $H$ $\mathfrak{F}$-sn $G$) if either $H = G$ or there exists a maximal chain of subgroups $H = H_0 \subset H_1 \subset \ldots \subset H_n = G$ with $H_i/Core_{H_i}(H_{i-1}) \in \mathfrak{F}$ for any $i \in \{1, \ldots, n\}$.

**Theorem 2.** *Let $\mathfrak{F}$ be a non-empty hereditary formation. Then $\mathrm{Int}_{\mathfrak{F}}(G/\mathrm{Int}_{\mathfrak{F}}(G)) \simeq 1$ holds for every group $G$ if and only if $\mathfrak{F}$ contains every group $G$ all of whose $\mathfrak{F}$-subgroups are $\mathfrak{F}$-subnormal in $G$.*

Recall that a group is called nilpotent if it coincides with its hypercenter. According to the results of D. W. Barnes and O. H. Kegel [15, Chapter IV, Proposition 1.5] if $\mathfrak{F}$ is a formation and $G \in \mathfrak{F}$, then $G$ coincides with its $\mathfrak{F}$-hypercenter. The class of all abelian groups shows that in the general case the converse of the previous statement fails. L. A. Shemetkov [16, Question 4.1] in 1997 posed the problem to describe a family of formations $\mathfrak{F} = (G \mid \mathrm{Z}_{\mathfrak{F}}(G) = G)$. Such formations are called $Z$-saturated [17].

**Theorem 3.** *Let $\mathfrak{F}$ be a hereditary formation of soluble groups. Then $\mathfrak{F}$ contains every group $G$ all of whose $\mathfrak{F}$-subgroups are $\mathfrak{F}$-subnormal in $G$ if and only if $\mathfrak{F}$ is a $Z$-saturated formation.*

**Corollary 1.** *Let $\mathfrak{F}$ be a non-empty hereditary formation of soluble groups. Then $\mathrm{Int}_{\mathfrak{F}}(G/\mathrm{Int}_{\mathfrak{F}}(G)) \simeq 1$ holds for every group $G$ if and only if $\mathfrak{F}$ is $Z$-saturated.*

**Example 1.** *Recall that the rank of a chief factor $H/K = H_1/K \times \ldots \times H_k/K$ in a group $G$ is the number $k$ of isomorphic simple groups $H_i/K$. The class of all soluble groups whose chief factor ranks do not exceed $n$ is a hereditary non-saturated $Z$-saturated formation for $n \geq 2$* [17, Theorem 6].

Classes of groups defined by systems of $\mathfrak{F}$-subnormal subgroups are widely studied nowadays (for example, see [18, 19, 20, 21, 22, 23, 24, 25]). In [19] and [21] the classes $v_\pi\mathfrak{F}$ and $w_\pi\mathfrak{F}$ of groups, whose all cyclic primary and primary $\pi$-subgroups are $\mathfrak{F}$-subnormal were studied respectively. The ideas of these works were combined in [18], where for a saturated homomorph $\mathfrak{H}$ and a hereditary formation $\mathfrak{F}$ the class $f_{\mathfrak{H}}(\mathfrak{F})$ of groups, all of whose $\mathfrak{H}$-subgroups are $\mathfrak{F}$-subnormal, was studied.

**Theorem 4.** *Let $\mathfrak{H}$ be a saturated homomorph, $\mathfrak{F} = f_{\mathfrak{H}}(\mathfrak{F})$ be a hereditary formation, $\mathfrak{H} \subseteq \mathfrak{F}$ and $G$ be a group. Then:*

(1) $\mathrm{Int}_{\mathfrak{F}}(G/N) = \mathrm{Int}_{\mathfrak{F}}(G)/N$ *for any* $N \trianglelefteq G$ *with* $N \leq \mathrm{Int}_{\mathfrak{F}}(G)$.
(2) $\mathrm{Int}_{\mathfrak{F}}(H)N/N \leq \mathrm{Int}_{\mathfrak{F}}(HN/N)$ *for any* $H \leq G$ *and* $N \trianglelefteq G$.
(3) *If* $L, S \leq G$, *then* $\mathrm{Int}_{\mathfrak{F}}(L) \cap S \leq \mathrm{Int}_{\mathfrak{F}}(L \cap S)$.

In [18] it was shown that the constructions $v_\pi\mathfrak{F}$ and $w_\pi\mathfrak{F}$ do not necessarily lead to saturated formations, i.e. hereditary $\check{S}$-formations have this property. For the applications of such formations, see [4, §24], [14, §6.4] and [26, §2.4].

**Corollary 2.** *Let $\mathfrak{F}$ be a non-empty hereditary formation with one of the following conditions:*

(*a*) $\mathfrak{F}$ *is a saturated formation* [9, Theorem C]*;*
(*b*) $\mathfrak{F} = v_\pi\mathfrak{F}$ *or* $\mathfrak{F} = w_\pi\mathfrak{F}$*;*
(*c*) $\mathfrak{F}$ *is a $\check{S}$-formation.*

*Then for every group $G$ the following statements hold:*

(1) $\mathrm{Int}_{\mathfrak{F}}(G/N) = \mathrm{Int}_{\mathfrak{F}}(G)/N$ *for any* $N \trianglelefteq G$ *with* $N \leq \mathrm{Int}_{\mathfrak{F}}(G)$.
(2) $\mathrm{Int}_{\mathfrak{F}}(H)N/N \leq \mathrm{Int}_{\mathfrak{F}}(HN/N)$ *for any* $H \leq G$ *and* $N \trianglelefteq G$.
(3) *If* $L, S \leq G$, *then* $\mathrm{Int}_{\mathfrak{F}}(L) \cap S \leq \mathrm{Int}_{\mathfrak{F}}(L \cap S)$.

Recall that the $\mathfrak{F}$-hypercenter is defined via the action of a group on its chief factors. The similar result for $\mathrm{Int}_{\mathfrak{F}}(G)$ was obtained in [11] when $\mathfrak{F}$ is a hereditary saturated formation and $G$ is a soluble group.

**Theorem 5.** *Let $\mathfrak{H}$ be a saturated homomorph, $\mathfrak{F} = f_{\mathfrak{H}}(\mathfrak{F})$ be a hereditary formation, $\mathfrak{H} \subseteq \mathfrak{F}$ and $H/K$ be a chief factor of a group $G$. Then $H/K \leq \mathrm{Int}_{\mathfrak{F}}(G/K)$ if and only if $H/K \rtimes TH/C_G(H/K) \in \mathfrak{F}$ for every $\mathfrak{H}$-subgroup $T/C_G(H/K)$ of $G/C_G(H/K)$.*

Theorems 4 and 5 give the $\mathfrak{F}$-hypercenter like characterization of $\mathrm{Int}_{\mathfrak{F}}(G)$ for hereditary formations $\mathfrak{F} = f_{\mathfrak{H}}(\mathfrak{F})$. In particular, in the case of saturated formations we get:

**Corollary 3.** *Let $\mathfrak{F}$ be a non-empty hereditary saturated formation, $f$ be its local definition and $G$ be a group. Then $\mathrm{Int}_{\mathfrak{F}}(G)$ is the largest normal subgroup of $G$ such that $HT/C_G(H/K) \in \mathfrak{N}_p f(p)$ for every $p \in \pi(H/K)$, chief factor $H/K$ of $G$ below it and $\mathfrak{F}$-subgroup $T/C_G(H/K)$ of $G/C_G(H/K)$.*

Since $H/K \leq C_G(H/K)$ for every chief factor $H/K$ of a soluble group $G$, from Corollary 3 we get:

**Corollary 4** ([11, Main Theorem])**.** *Let $\mathfrak{F}$ be a non-empty hereditary saturated formation, $f$ be a local definition of $\mathfrak{F}$, $G$ be a soluble group and*

$$g(p) = (L \mid \mathrm{O}_p(L) = 1 \text{ and every } \mathfrak{F}\text{-subgroup of } L \text{ belongs to } \mathfrak{N}_p f(p)).$$

*Then $\mathrm{Int}_{\mathfrak{F}}(G)$ is the largest among normal subgroups $M$ of $G$ with the following property: if $H/K$ is a $p$-chief factor of $G$ with $H \subseteq M$, then $G/C_G(H/K) \in g(p)$.*

The computational aspects of these results will be discussed in the last section.

## 2 Preliminaries

The notation and terminology agree with [14, 15]. Let us recall some of them: $Z_n$ is the cyclic group of order $n$; $S_n$ and $A_n$ are the symmetric and the alternating groups of degree $n$ respectively; $Core_G(H)$ is the largest normal subgroup of the group $G$ that is contained in $H$; if $M$ is a proper subgroup with the property that $M = H$ whenever $M \leq H < G$, then $M$ is called a maximal subgroup of $G$; a chain of subgroups $H = H_0 \subset H_1 \subset \ldots \subset H_n = G$ is called maximal in $G$ if either $n = 0$ or $H_{i-1}$ is a maximal subgroup of $H_i$ for any $i > 0$; a group $G \in \mathfrak{F}$ is called an $\mathfrak{F}$-group; a subgroup $U$ of $G$ is called $\mathfrak{F}$-maximal in $G$ provided $U \in \mathfrak{F}$ and if $U \leq V \leq G$, $V \in \mathfrak{F}$, then $U = V$; $\Phi(G)$ is the Frattini subgroup of $G$, i.e. the intersection of all maximal subgroups of $G$; $\mathrm{F}(G)$ is the Fitting subgroup of $G$, i.e. the largest normal nilpotent subgroup of $G$; $N \rtimes M$ is the semidirect product of $N$ with $M$; $\mathrm{Aut}(G)$ is the group of all automorphisms of a group $G$; $G^{\mathfrak{F}}$ is the $\mathfrak{F}$-residual of $G$, i.e. the intersection of all those normal subgroups $N$ of $G$ for which $G/N \in \mathfrak{F}$; $G_{\mathfrak{S}}$ is the soluble radical of $G$, i.e. the largest normal soluble subgroup of $G$.

Recall [15, Chapter II, Definition 1.1] that a class of groups is a collection $\mathfrak{X}$ of groups with the property that if $G \in \mathfrak{X}$ and if $H \simeq G$, then $H \in \mathfrak{X}$. The classes of all soluble groups and of all $p$-groups for a prime $p$ are denoted by $\mathfrak{S}$ and $\mathfrak{N}_p$, respectively. A class of groups $\mathfrak{H}$ is called a homomorph if $G \in \mathfrak{H}$ and $N \trianglelefteq G$, then $G/N \in \mathfrak{H}$. A formation is a homomorph $\mathfrak{F}$ that is closed under taking subdirect products, i.e. from $H/A \in \mathfrak{F}$, $H/B \in \mathfrak{F}$ it follows that $H/(A \cap B) \in \mathfrak{F}$. A class of groups $\mathfrak{F}$ is called: hereditary if $G \in \mathfrak{F}$ and $H \leq G$, then $H \in \mathfrak{F}$; saturated if $G/N \in \mathfrak{F}$ and $N \leq \Phi(G)$, then $G \in \mathfrak{F}$; solubly saturated if $G/N \in \mathfrak{F}$ and $N \leq \Phi(G_{\mathfrak{S}})$, then $G \in \mathfrak{F}$. The formation product of $\mathfrak{F}_1$ with $\mathfrak{F}_2$ is defined as $\mathfrak{F}_1\mathfrak{F}_2 = (G \mid G^{\mathfrak{F}_2} \in \mathfrak{F}_1)$.

Recall that a minimal non-$\mathfrak{F}$-group for a class of groups $\mathfrak{F}$ is a group $G \notin \mathfrak{F}$, all of whose proper subgroups belong to $\mathfrak{F}$; a Schmidt group is a minimal non-nilpotent group; a formation $\mathfrak{F}$ is called a $\check{S}$-formation if every minimal non-$\mathfrak{F}$-group is either a Schmidt group or a group of prime order (see [14, §6.4]).

**Theorem 6** (D. W. Barnes and O. H. Kegel [15, Chapter IV, Proposition 1.5])**.** *Let $\mathfrak{F}$ be a formation, $R/S$ be a normal section of a group $G \in \mathfrak{F}$ and $K$ be a normal subgroup of $G$ contained in $C_G(R/S)$. With respect to the following action of $G/K$ on $R/S$: $(rS)^{gK} = g^{-1}rgS$, $r \in R, g \in G$, form the semidirect product $H = R/S \rtimes G/K$. Then $H \in \mathfrak{F}$.*

**Lemma 1** ([14, Lemma 6.1.6])**.** *Let $\mathfrak{F}$ be a formation, $G$ be a group, $H, R \leq G$ and $N \trianglelefteq G$. Then:*

(1) *If $H/N$ $\mathfrak{F}$-sn $G/N$, then $H$ $\mathfrak{F}$-sn $G$.*
(2) *If $H$ $\mathfrak{F}$-sn $G$, then $HN/N$ $\mathfrak{F}$-sn $G/N$.*
(3) *If $H$ $\mathfrak{F}$-sn $R$ and $R$ $\mathfrak{F}$-sn $G$, then $H$ $\mathfrak{F}$-sn $G$.*

The next result directly follows from Lemma 1:

**Lemma 2.** *Let $\mathfrak{F}$ be a formation, $G$ be a group, $H, R \leq G$ and $N \trianglelefteq G$. If $H$ $\mathfrak{F}$-sn $R$, then $HN$ $\mathfrak{F}$-sn $RN$.*

Recall [14, 26] that a subgroup functor is a function $f$ which assigns to each group $G$ a possibly empty set $f(G)$ of subgroups of $G$ satisfying $\theta(f(G)) = f(\theta(G))$ for any isomorphism $\theta : G \to G^*$. The function $s_{n\mathfrak{F}}$ that assigns to each group the

set of its $\mathfrak{F}$-subnormal subgroups is a subgroup functor [14, 26]. The next result directly follows from this fact.

**Lemma 3.** *Let $G$ be a group and $H, R \leq G$. If $\mathfrak{F}$ is a non-empty formation and $H$ $\mathfrak{F}$-sn $R$, then $H^\alpha$ $\mathfrak{F}$-sn $R^\alpha$ for all $\alpha \in \mathrm{Aut}(G)$.*

**Lemma 4** ([14, Lemma 6.1.7])**.** *Let $\mathfrak{F}$ be a hereditary formation, $G$ be a group and $H, R \leq G$. If $H$ $\mathfrak{F}$-sn $G$, then $H \cap R$ $\mathfrak{F}$-sn $R$.*

A function $f$, which assigns to every prime a formation, is called a formation function. Recall [15, Chapter IV, Definition 3.1] that a formation $\mathfrak{F}$ is called local if

$$\mathfrak{F} = (G \mid G/C_G(H/K) \in f(p) \text{ for every } p \in \pi(H/K) \text{ and chief factor } H/K \text{ of } G)$$

for some formation function $f$. In this case $f$ is called a local definition of $\mathfrak{F}$. According to the Gaschütz – Lubeseder – Schmid theorem [15, Chapter IV, Theorem 4.6] a non-empty formation is saturated iff it is local.

Recall [17, Theorem 1] that $Z\mathfrak{F} = (G \mid G = \mathrm{Z}_{\mathfrak{F}}(G))$ is a $Z$-saturated formation for a class $\mathfrak{F}$ of groups. From this fact and from [27, Lemma 2.4] the following result is immediately obtained:

**Lemma 5.** *Let $\mathfrak{F}$ be a hereditary formation. Then $Z\mathfrak{F}$ is a hereditary $Z$-saturated formation.*

Recall that a supernatural number is the following formal product $\omega = \prod_{p\in\mathbb{P}} p^{v_p(\omega)}$ where $v_p(\omega)$ can be zero, a natural number or infinity. For supernatural numbers $\omega_1$ and $\omega_2$, recall that $\mathrm{lcm}(\omega_1, \omega_2) = \prod_{p\in\mathbb{P}} p^{\max\{v_p(\omega_1),\, v_p(\omega_2)\}}$ and $\omega_1$ divides $\omega_2$ if $v_p(\omega_1) \leq v_p(\omega_2)$ for all $p \in \mathbb{P}$. The exponent of a class of groups $\mathfrak{X}$ is defined by $exp(\mathfrak{X}) = \mathrm{lcm}\{exp(G) \mid G \in \mathfrak{X}\}$.

## 3 The intersection of weak $\mathfrak{F}$-subnormalizers

R. W. Carter [28] and C. J. Graddon [29] studied subnormalizers and $\mathfrak{F}$-subnormalizers respectively. A subnormalizer or an $\mathfrak{F}$-subnormalizer may not exist in a group for a given subgroup. A. Mann [30] proposed the concept of a weak subnormalizer, which always exists, but may not be unique. In [13] the concept of a weak K-$\mathfrak{F}$-subnormalizer was proposed. By analogy we propose the following definition:

**Definition 1.** *We shall call a subgroup $T$ of $G$ a weak $\mathfrak{F}$-subnormalizer of $H$ in $G$ if $H$ $\mathfrak{F}$-sn $T$ and if $H$ $\mathfrak{F}$-sn $M \leq G$ and $T \leq M$, then $T = M$.*

Let $\mathfrak{F}$ be a hereditary formation and $H \leq G$. From the definitions of $\mathfrak{F}$-subnormal subgroup and weak $\mathfrak{F}$-subnormalizer it follows that $H$ $\mathfrak{F}$-sn $G$ if and only if $G$ coincides with the weak $\mathfrak{F}$-subnormalizer of $H$ in $G$.

**Lemma 6.** *Let $\mathfrak{F}$ be a non-empty formation, $G$ be a group and $H \leq G$. If $T$ is a weak $\mathfrak{F}$-subnormalizer of $H$ in $G$, then $T^\alpha$ is a weak $\mathfrak{F}$-subnormalizer of $H^\alpha$ in $G$ for any $\alpha \in \mathrm{Aut}(G)$.*

*Proof.* Assume that $T$ is a weak $\mathfrak{F}$-subnormalizer of $H$ and $T^\alpha$ is not a weak $\mathfrak{F}$-subnormalizer of $H^\alpha$ for some $\alpha \in \mathrm{Aut}(G)$. Since $H^\alpha$ $\mathfrak{F}$-sn $T^\alpha$ by Lemma 3, there exists $M \leq G$ such that $H^\alpha$ $\mathfrak{F}$-sn $M$ and $T^\alpha < M$. Then $H$ $\mathfrak{F}$-sn $M^{\alpha^{-1}}$ by Lemma 3 and $T < M^{\alpha^{-1}}$, a contradiction. Therefore $T^\alpha$ is a weak $\mathfrak{F}$-subnormalizer of $H^\alpha$ for any $\alpha \in \mathrm{Aut}(G)$. □

Our main idea is to study $\mathrm{Int}_{\mathfrak{F}}(G)$ using the following subgroup:

**Definition 2.** *Let $\mathfrak{F}$ be a formation, $\mathfrak{H}$ be a homomorph and $G$ be a group. By $\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)$ we shall denote the intersection of all weak $\mathfrak{F}$-subnormalizers of all $\mathfrak{H}$-subgroups of $G$.*

**Lemma 7.** *Let $\mathfrak{H}$ be a saturated homomorph. If $A/N \in \mathfrak{H}$ and $B$ is the minimal supplement to $N$ in $A$, then $B \in \mathfrak{H}$ and $BN/N = A/N$.*

*Proof.* Note that $B \cap N \subseteq \Phi(B)$ by [15, Chapter A, Theorem 9.2]. Since $A/N = BN/N \simeq B/(B \cap N) \in \mathfrak{H}$, we obtain that $B \in \mathfrak{H}$. □

**Theorem 7.** *Let $\mathfrak{F}$ be a hereditary formation, $\mathfrak{H}$ be a homomorph, $G$ be a group, $N \trianglelefteq G$ and $L \leq G$. The following statements hold:*

(1) *$\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)$ is the characteristic subgroup of $G$ and is the largest subgroup among normal subgroups $N_1$ of $G$ with $H$ $\mathfrak{F}$-sn $HN_1$ for any $\mathfrak{H}$-subgroup $H$ of $G$.*

(2) *If $\mathfrak{H}$ is saturated, then $\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(L)N/N \leq \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(LN/N)$.*

(3) *If $N \leq \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)$, then $\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G/N) \leq \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)/N$, i.e. $\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G/\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)) \simeq 1$. If $\mathfrak{H}$ is saturated, then $\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G/N) = \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)/N$.*

(4) *If $S \leq G$, then $\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(L) \cap S \leq \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(L \cap S)$.*

(5) *$\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)) = \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)$.*

*Proof.* (1) Let $\mathcal{K}$ be the set of all weak $\mathfrak{F}$-subnormalizers of $\mathfrak{H}$-subgroups of $G$ and $T \in \mathcal{K}$. Then $T$ is a weak $\mathfrak{F}$-subnormalizer of some $\mathfrak{H}$-subgroup $H$. Since $\mathfrak{H}$ is a class of groups, $H^{\alpha} \in \mathfrak{H}$ for any $\alpha \in \mathrm{Aut}(G)$. Then $T^{\alpha} \in \mathcal{K}$ for any $\alpha \in \mathrm{Aut}(G)$ by Lemma 6. Note that

$$(\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G))^{\alpha} = (\cap_{T \in \mathcal{K}} T)^{\alpha} = (\cap_{T \in \mathcal{K}} T^{\alpha^{-1}})^{\alpha} = \cap_{T \in \mathcal{K}} T^{\alpha^{-1}\alpha} = \cap_{T \in \mathcal{K}} T = \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)$$

for any $\alpha \in \mathrm{Aut}(G)$. Therefore $\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)$ is the characteristic subgroup of $G$.

Let $H$ be an $\mathfrak{H}$-subgroup of $G$ and $N_1$ be a normal subgroup of $G$ such that $H_1$ $\mathfrak{F}$-sn $H_1N_1$ for any $\mathfrak{H}$-subgroup $H_1$ of $G$. If $T$ is a weak $\mathfrak{F}$-subnormalizer of $H$ in $G$, then $HN_1$ $\mathfrak{F}$-sn $TN_1$ by Lemma 2. Hence $H$ $\mathfrak{F}$-sn $TN_1$ by (3) of Lemma 1. So $TN_1 = T$ by the definition of a weak $\mathfrak{F}$-subnormalizer. Thus $N_1 \leq \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)$.

On the other hand, since $\mathfrak{F}$ is a hereditary formation and $H\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)$ is contained in every weak $\mathfrak{F}$-subnormalizer of $H$ in $G$, it follows that $H$ $\mathfrak{F}$-sn $H\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)$ by Lemma 4. Then $\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)$ is the largest subgroup among normal subgroups $N_1$ of $G$ with $H_1$ $\mathfrak{F}$-sn $H_1N_1$ for any $\mathfrak{H}$-subgroup $H_1$ of $G$.

(2) Let $L = G$ and $I = \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)$. For any $V/N \in \mathfrak{H}$ there exists $L_1 \in \mathfrak{H}$ with $L_1N = V$ by Lemma 7. So $L_1$ $\mathfrak{F}$-sn $L_1I$ by (1). Therefore $L_1N/N = V/N$ $\mathfrak{F}$-sn $VI/N$ by Lemma 1. Hence $V/N$ $\mathfrak{F}$-sn $VI/N$ for any $V/N \in \mathfrak{H}$. It follows that $\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)N/N \leq \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G/N)$ by (1).

Now let $L$ be any subgroup of $G$ and let $f : L/(L \cap N) \to LN/N$ be the canonical isomorphism. Then

$$f(\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(L/(L \cap N))) = \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(LN/N) \text{ and } f(\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(L)(L \cap N)/(L \cap N)) = \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(L)N/N$$

by (1). From our previous results we get

$$\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(L)(L \cap N)/(L \cap N) \leq \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(L/(L \cap N)).$$

Hence $\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(L)N/N \leq \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(LN/N)$.

(3) Let $N \leq \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)$ and $\overline{K} = \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G/N) = K/N$. Note that $K \trianglelefteq G$. Let $H \leq G$ and $H \in \mathfrak{H}$. Hence $\overline{H} = HN/N \simeq H/(H \cap N) \in \mathfrak{H}$. Let $\overline{T}$ be a weak $\mathfrak{F}$-subnormalizer of $\overline{H}$ in $G/N$, i.e. $\overline{H}$ $\mathfrak{F}$-sn $\overline{T}$. From $\overline{H} \leq \overline{KH} \leq \overline{T}$ it follows that $\overline{H} = HN/N$ $\mathfrak{F}$-sn $\overline{KN} = HK/N$. So $HN$ $\mathfrak{F}$-sn $HK$ by (2) of Lemma 1. Since $H$ $\mathfrak{F}$-sn $HN$ by Lemma 4, we obtain that $H$ $\mathfrak{F}$-sn $HK$ by (3) of Lemma 1. Let $T_1$ be a weak $\mathfrak{F}$-subnormalizer of $H$ in $G$. Then $HK$ $\mathfrak{F}$-sn $T_1K$ by Lemma 2. Since $H$ $\mathfrak{F}$-sn $HK$ and $HK$ $\mathfrak{F}$-sn $T_1K$, $H$ $\mathfrak{F}$-sn $T_1K$ by (3) of Lemma 1. So $T_1K = T_1$ by the definition of weak $\mathfrak{F}$-subnormalizer. Therefore $K$ lies in all weak $\mathfrak{F}$-subnormalizers of all $\mathfrak{H}$-subgroups of $G$. Hence $K \leq I$, i.e. $\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G/N) \leq \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)/N$.

If $N = \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)$, then $N/N \leq \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G/N) \leq N/N$. So $\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G/\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)) \simeq 1$.

If $\mathfrak{H}$ is a saturated homomorph, then $\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)/N \leq \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G/N)$ by (2). Therefore $\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)/N = \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G/N)$.

(4) Note that $K$ $\mathfrak{F}$-sn $K\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(L)$ for any $\mathfrak{H}$-subgroup $K$ of $L \cap S$ by (1) and $K(\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(L) \cap S) \leq K\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(L)$. So $K$ $\mathfrak{F}$-sn $K(\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(L) \cap S)$ for any $\mathfrak{H}$-subgroup $K$ of $L \cap S$. Thus $\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(L) \cap S \leq \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(L \cap S)$ by (1).

(5) From (4) it follows that

$$\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)) \leq \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G) = \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G) \cap \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G) \leq \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G \cap \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)) = \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)).$$

Thus $\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)) = \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)$. □

The key idea in the proof of the main results of the work is:

**Theorem 8.** *Let $\mathfrak{H}$ be a homomorph and $\mathfrak{F}$ be a hereditary formation with $\mathfrak{H} \subseteq \mathfrak{F}$. Then $\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G) = \mathrm{Int}_{\mathfrak{F}}(G)$ holds for every group $G$ if and only if $\mathfrak{F} = f_{\mathfrak{H}}(\mathfrak{F})$.*

*Proof.* Assume that $\mathfrak{F} = f_{\mathfrak{H}}(\mathfrak{F})$. Then $1 \in \mathfrak{F}$. So $\mathfrak{F} \neq \emptyset$. Let $M$ be an $\mathfrak{F}$-maximal subgroup of $G$, $I = \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)$ and $H$ be an $\mathfrak{H}$-subgroup of $MI$. Then $H$ $\mathfrak{F}$-sn $HI$ by (1) of Theorem 7. Note that $MI/I \simeq M/(M \cap I) \in \mathfrak{F}$. Since $\mathfrak{F}$ is a hereditary formation, $HI/I$ $\mathfrak{F}$-sn $MI/I$. Hence $HI$ $\mathfrak{F}$-sn $MI$ by (1) of Lemma 1. Thus $H$ $\mathfrak{F}$-sn $MI$ for any $\mathfrak{H}$-subgroup $H$ of $MI$. Therefore $MI \in \mathfrak{F}$ by our assumption. Hence $MI = M$. So $I \subseteq \mathrm{Int}_{\mathfrak{F}}(G)$.

Let $H$ be an $\mathfrak{H}$-subgroup of $G$. Then from $\mathfrak{H} \subseteq \mathfrak{F}$ it follows that $H$ is contained in some $\mathfrak{F}$-maximal subgroup of $G$. So $H\mathrm{Int}_{\mathfrak{F}}(G) \in \mathfrak{F}$. Hence $H$ $\mathfrak{F}$-sn $H\mathrm{Int}_{\mathfrak{F}}(G)$ for any $\mathfrak{H}$-subgroup $H$ of $G$. Now $\mathrm{Int}_{\mathfrak{F}}(G) \subseteq I$ by (1) of Theorem 7. Thus $I = \mathrm{Int}_{\mathfrak{F}}(G)$.

Assume that $\mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G) = \mathrm{Int}_{\mathfrak{F}}(G)$ holds for every group $G$. Since $\mathfrak{F}$ is a hereditary formation, it is obvious that $\mathfrak{F} \subseteq f_{\mathfrak{H}}(\mathfrak{F})$. Let $G \in f_{\mathfrak{H}}(\mathfrak{F})$. Then $G = \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)$. Hence $G = \mathrm{Int}_{\mathfrak{F}}(G)$ by our assumption. It means that $G \in \mathfrak{F}$. Therefore $f_{\mathfrak{H}}(\mathfrak{F}) \subseteq \mathfrak{F}$. Thus $\mathfrak{F} = f_{\mathfrak{H}}(\mathfrak{F})$. □

## 4 Proofs of the main results

*Proof of Theorem 1.* Let $\mathfrak{F}_1$ be a class of groups, all of whose composition factors are from $(Z_2, Z_3, Z_5, A_5)$. It is clear that this class is a saturated formation and therefore it is solubly saturated. Recall [5, Chapter 1, Definition 3.1] that a group $G$ is called quasisoluble if for every $\mathfrak{S}$-eccentric (i.e. non $\mathfrak{S}$-central) chief factor $H/K$ of $G$, every automorphism of $H/K$ induced by an element of $G$ is inner. Let $\mathfrak{F}_2$ be the class of all quasisoluble groups. Then $\mathfrak{F}_2$ is a solubly saturated formation by [5, Chapter 1, Corollary 3.8].

Let $\mathfrak{F} = \mathfrak{F}_1 \cap \mathfrak{F}_2$. Then it is solubly saturated as the intersection of solubly saturated formations. So $G/\mathrm{Z}_{\mathfrak{S}}(G)$ is a direct product of groups isomorphic to $A_5$ for any $\mathfrak{F}$-group $G$ by [5, Chapter 1, Theorem 3.12(2)]. Let $H$ be a subgroup of an $\mathfrak{F}$-group $G$. Then $H\mathrm{Z}_{\mathfrak{S}}(G)/\mathrm{Z}_{\mathfrak{S}}(G)$ is a subgroup of the direct product of groups isomorphic to $A_5$. Since all proper subgroups of $A_5$ and $A_5$ itself belong to $\mathfrak{F}$, the projection of $H\mathrm{Z}_{\mathfrak{S}}(G)/\mathrm{Z}_{\mathfrak{S}}(G)$ onto each of the factors of the direct product belongs to $\mathfrak{F}$. Note that $\mathfrak{F}$ is closed under taking subdirect products and the intersection of the kernels of the given projections is trivial. Hence $H\mathrm{Z}_{\mathfrak{S}}(G)/\mathrm{Z}_{\mathfrak{S}}(G) \in \mathfrak{F}$. Note that $H\mathrm{Z}_{\mathfrak{S}}(G)/\mathrm{Z}_{\mathfrak{S}}(G) \simeq H/(\mathrm{Z}_{\mathfrak{S}}(G) \cap H)$ and $\mathrm{Z}_{\mathfrak{S}}(G) \cap H \subseteq \mathrm{Z}_{\mathfrak{S}}(H)$ by [5, Chapter 1, Theorem 2.7(a)]. Since the class of soluble $\{2,3,5\}$-groups is contained in $\mathfrak{F}$, the soluble hypercenter of $H$ lies in the $\mathfrak{F}$-hypercenter of $H$. So $H \in \mathfrak{F}$. Thus $\mathfrak{F}$ is a hereditary formation.

According to the result of R. L. Griess and P. Schmid [15, Appendix $\beta$, $\beta$.9] for $p = 2$ and $G \simeq A_5$ there exists a Frattini $\mathbb{F}_2G$-module $A$ which is faithful for $G$. By the well-known theorem of W. Gaschütz [15, Appendix $\beta$] there exists a Frattini extension $A \rightarrowtail R \twoheadrightarrow G$ such that $A \simeq \Phi(R)$ and $R/\Phi(R) \simeq G$. Note that all maximal subgroups of $G$ are soluble and the class of soluble groups is closed under extensions. Hence all maximal subgroups of $R$ are soluble. Since $A$ is a faithful module, there exists a chief factor $H/K$ of $R$ such that $R/C_R(H/K) = G$ is not soluble. Now $H/K$ is an abelian noncentral chief factor of $R$. Hence $R$ does not act by inner automorphisms on it. Therefore $R \notin \mathfrak{F}$. So $\mathrm{Int}_{\mathfrak{F}}(R) = A$ but $R/A = G \in \mathfrak{F}$. Thus $\mathrm{Int}_{\mathfrak{F}}(R/\mathrm{Int}_{\mathfrak{F}}(R)) \not\simeq 1$. □

*Proof of Theorem 2.* Assume that $\mathrm{Int}_{\mathfrak{F}}(G/\mathrm{Int}_{\mathfrak{F}}(G)) \simeq 1$ holds for any group $G$ and there exists a non-$\mathfrak{F}$-group all of whose $\mathfrak{F}$-subgroups are $\mathfrak{F}$-subnormal. Let $G$ be a minimal order group with this property. Since $\mathfrak{F}$ is a hereditary formation, all $\mathfrak{F}$-subgroups of every maximal subgroup of $G$ are $\mathfrak{F}$-subnormal in it by Lemma 4. Hence $M \in \mathfrak{F}$ for every maximal subgroup $M$ of $G$ by our assumption. Therefore $\mathrm{Int}_{\mathfrak{F}}(G) = \Phi(G)$. Assume that $\Phi(G) \neq 1$. If $G/\Phi(G) \in \mathfrak{F}$, then

$$1 \simeq \mathrm{Int}_{\mathfrak{F}}(G/\mathrm{Int}_{\mathfrak{F}}(G)) = \mathrm{Int}_{\mathfrak{F}}(G/\Phi(G)) = G/\Phi(G),$$

a contradiction. Thus $G/\Phi(G) \notin \mathfrak{F}$. Let $H/\Phi(G)$ be a proper $\mathfrak{F}$-subgroup of $G/\Phi(G)$. Then $H$ is a proper subgroup of $G$, i.e. $H \in \mathfrak{F}$. Thus $H/\Phi(G)$ $\mathfrak{F}$-sn $G/\Phi(G)$ by (2) of Lemma 1. It means that $G/\Phi(G) \in \mathfrak{F}$ by our assumption, a contradiction. Thus $\Phi(G) = 1$. Since $1$ $\mathfrak{F}$-sn $G$, we obtain that $G^{\mathfrak{F}} < G$. From $\Phi(G) = 1$ it follows that there is a maximal subgroup $M$ of $G$ with $G^{\mathfrak{F}} \not\leq M$. Since $M$ is $\mathfrak{F}$-subnormal and maximal in $G$, it follows that $G^{\mathfrak{F}} \leq M$, the final contradiction.

Assume that $\mathfrak{F}$ contains every group $G$ all of whose $\mathfrak{F}$-subgroups are $\mathfrak{F}$-subnormal in $G$. Then $\mathrm{Int}_{\mathfrak{F}}(G/\mathrm{Int}_{\mathfrak{F}}(G)) \simeq 1$ holds for every group $G$ by Theorems 7 and 8. □

**Lemma 8.** *Let $\mathfrak{H}$ be a homomorph and $\mathfrak{F} = f_{\mathfrak{H}}(\mathfrak{F})$ be a hereditary formation. Then $\mathfrak{F}$ is a hereditary $Z$-saturated formation.*

*Proof.* Note that $Z\mathfrak{F}$ is a hereditary $Z$-saturated formation by Lemma 5. Assume that $\mathfrak{F}$ is not $Z$-saturated. Hence $\mathfrak{F} \subset Z\mathfrak{F}$. Let $G$ be a minimal order group from $Z\mathfrak{F} \setminus \mathfrak{F}$. Then all proper subgroups and quotient groups of $G$ belong to $\mathfrak{F}$. Since $\mathfrak{F}$ is a formation, $G$ has the unique minimal normal subgroup $N$. If $N$ is non-abelian, then $G/C_G(N) \simeq G$. From $N \rtimes (G/C_G(N)) \in \mathfrak{F}$ it follows that $G \in \mathfrak{F}$, a contradiction. So $N$ is abelian. If $G$ is a primitive group, then $G \simeq N \rtimes G/C_G(N) \in \mathfrak{F}$ by [14,

Proposition 1.1.12], a contradiction. Hence $G$ is not a primitive group. Therefore $N \leq \Phi(G)$.

If $G \in \mathfrak{H}$, then $G\,\mathfrak{F}\text{-sn}\,G$. Let $H$ be a proper $\mathfrak{H}$-subgroup of $G$. Then $HN < G$. Hence $HN \in \mathfrak{F}$. So $H\,\mathfrak{F}\text{-sn}\,HN$. From $HN/N \simeq H/(H \cap N) \in \mathfrak{H}$ and $G/N \in \mathfrak{F} = f_{\mathfrak{H}}(\mathfrak{F})$ it follows that $HN/N\,\mathfrak{F}\text{-sn}\,G/N$. So $HN\,\mathfrak{F}\text{-sn}\,G$ by Lemma 1. Now $H\,\mathfrak{F}\text{-sn}\,G$ by Lemma 1. Thus $G \in f_{\mathfrak{H}}(\mathfrak{F}) = \mathfrak{F}$, the final contradiction. □

*Proof of Theorem 3.* Let $\mathfrak{F} = f_{\mathfrak{F}}(\mathfrak{F})$. Since $\mathfrak{F}$ is a homomorph, $\mathfrak{F}$ is a $Z$-saturated formation by Lemma 8.

Let $\mathfrak{F}$ be a $Z$-saturated formation. It is clear that $(1) \subseteq \mathfrak{F} \subseteq f_{\mathfrak{F}}(\mathfrak{F})$. Assume that $\mathfrak{F} \neq f_{\mathfrak{F}}(\mathfrak{F})$. Let $G$ be a minimal order group from $f_{\mathfrak{F}}(\mathfrak{F}) \setminus \mathfrak{F}$. Then all maximal subgroups of $G$ belong to $\mathfrak{F}$ by Lemma 4 and our assumption. Since $1\ \mathfrak{F}\text{-sn}\ G$ and $\mathfrak{F} \subseteq \mathfrak{S}$, we see that $G \in \mathfrak{S}$. Note that if all subgroups of $G$ are $\mathfrak{F}$-subnormal, then all subgroups of $G/N$ are also $\mathfrak{F}$-subnormal by (2) of Lemma 1. If $|N| > 1$, then $G/N \in \mathfrak{F}$ by assumption. So $G$ has a unique minimal normal subgroup $N$ and $G/N \in \mathfrak{F}$. Thus $N = G^{\mathfrak{F}}$. Since $M\ \mathfrak{F}\text{-sn}\ G$, we obtain that $N \leq M$ for every maximal subgroup $M$ of $G$. Therefore $N \leq \Phi(G)$. Since $G$ is soluble, $\Phi(G) < \mathrm{F}(G)$ and $\mathrm{F}(G) \leq C_G(N)$ [15, Chapter A, Theorem 10.6(b)]. Let $M$ be a maximal subgroup of $G$ such that $M\mathrm{F}(G) = G$. Then $G = MC_G(N)$. Hence $N$ is a minimal normal subgroup of $M$. Since $\mathfrak{F}$ is a formation and $M \in \mathfrak{F}$, we have $N \rtimes G/C_G(N) \simeq N \rtimes M/C_M(N) \in \mathfrak{F}$ by Theorem 6. Thus $N$ is an $\mathfrak{F}$-central chief factor of $G$. Since $G/N \in \mathfrak{F}$ and $\mathfrak{F}$ is $Z$-saturated, we see that $G \in \mathfrak{F}$, the final contradiction. □

*Proof of Theorem 4.* If $\mathfrak{F} = f_{\mathfrak{H}}(\mathfrak{F})$, then $1 \in \mathfrak{F}$. Note that $\mathrm{Int}_{\mathfrak{F}}(G) = \mathcal{I}^{\mathfrak{F}}_{\mathfrak{H}}(G)$ holds for every group $G$ by Theorem 8. Now statements (1), (2), (3) of Theorem 4 directly follow from statements (2), (3), (4) of Theorem 7. □

**Lemma 9.** *Let $\mathfrak{F}$ be a non-empty hereditary saturated formation. Then $\mathfrak{F} = f_{\mathfrak{F}}(\mathfrak{F})$.*

*Proof.* Since $\mathfrak{F}$ is hereditary, it is clear that $\mathfrak{F} \subseteq f_{\mathfrak{F}}(\mathfrak{F})$. Assume that $f_{\mathfrak{F}}(\mathfrak{F}) \setminus \mathfrak{F} \neq \emptyset$. Let $G$ be a minimal order group from $f_{\mathfrak{F}}(\mathfrak{F}) \setminus \mathfrak{F}$. Hence all proper subgroups and quotient groups of $G$ belong to $\mathfrak{F}$ by Lemmas 1 and 4. Let $M$ be a maximal subgroup of $G$. Then $M \in \mathfrak{F}$ and $MG^{\mathfrak{F}} < G$. Hence $G^{\mathfrak{F}} \leq M$. Therefore $G^{\mathfrak{F}} \leq \Phi(G)$. From $G \notin \mathfrak{F}$ it follows that $\Phi(G) \neq 1$. Now $G/\Phi(G) \in \mathfrak{F}$. Since $\mathfrak{F}$ is saturated, we see that $G \in \mathfrak{F}$, the contradiction. Thus $\mathfrak{F} = f_{\mathfrak{F}}(\mathfrak{F})$. □

*Proof of the Corollary 2.* (*a*) Let $\mathfrak{H} = \mathfrak{F} \subseteq \mathfrak{F}$. Then $\mathfrak{F} = f_{\mathfrak{F}}(\mathfrak{F})$ by Lemma 9. Therefore (a) of Corollary 2 follows from Theorem 4.

(*b*) Let $\mathfrak{F} = v_\pi\mathfrak{F}$. Then $\mathfrak{F} = v_{\pi\cap\pi(\mathfrak{F})}\mathfrak{F}$ by [19, Theorem A(2)]. Let $\mathfrak{H} = (G \mid G$ is a cyclic primary $(\pi \cap \pi(\mathfrak{F}))$-group). Since $\mathfrak{H} \subseteq \mathfrak{N}_{\pi\cap\pi(\mathfrak{F})} \subseteq \mathfrak{F}$ by [19, Theorem A(1)], we have that the case $\mathfrak{F} = v_\pi\mathfrak{F}$ of Corollary 2 follows from Theorem 4.

Let $\mathfrak{F} = w_\pi\mathfrak{F}$. Then $\mathfrak{F} = w_{\pi\cap\pi(\mathfrak{F})}\mathfrak{F}$ by [21, Theorem 3.1(4)]. Let $\mathfrak{H} = (G \mid G$ is a primary $(\pi\cap\pi(\mathfrak{F}))$-group). Since $\mathfrak{H} \subseteq \mathfrak{N}_{\pi\cap\pi(\mathfrak{F})} \subseteq \mathfrak{F}$ by [21, Theorem 3.1(3)], we have that the case $\mathfrak{F} = w_\pi\mathfrak{F}$ of Corollary 2 follows from Theorem 4.

(*c*) Since $\mathfrak{F} = v\mathfrak{F}$ by [18, Corollary 3.9], (*c*) follows from (*b*). □

*Proof of Theorem 5.* If $\mathfrak{F} = f_{\mathfrak{H}}(\mathfrak{F})$, then $1 \in \mathfrak{F}$. Assume that $H/K \leq \mathrm{Int}_{\mathfrak{F}}(G/K)$. Note that $C_G(H/K)/K = C_{G/K}(H/K)$. If

$$T/C_G(H/K) \simeq (T/K)/(C_G(H/K)/K) = (T/K)/C_{G/K}(H/K) \in \mathfrak{H},$$

then by Lemma 7 there exists $L/K \in \mathfrak{H}$ such that

$$(L/K)C_{G/K}(H/K)/C_{G/K}(H/K) \simeq (T/K)/C_{G/K}(H/K).$$

From $\mathfrak{H} \subseteq \mathfrak{F}$ it follows that there is an $\mathfrak{F}$-maximal subgroup $M/K$ of $G/K$ with $L/K \leq M/K$. Since $H \leq M$ and $\mathfrak{F}$ is hereditary, we see that $LH/K \in \mathfrak{F}$. Hence $H/K \rtimes (LH/K)/C_{LH/K}(H/K) \in \mathfrak{F}$ by Theorem 6. Therefore

$$H/K \rtimes (LH/K)/C_{LH/K}(H/K) = H/K \rtimes (LH/K)/(C_{G/K}(H/K) \cap (LH/K)) \simeq$$
$$H/K \rtimes (LH/K)C_{G/K}(H/K)/C_{G/K}(H/K) \simeq$$
$$H/K \rtimes (LHC_G(H/K)/K)/(C_G(H/K)/K) \simeq$$
$$H/K \rtimes LHC_G(H/K)/C_G(H/K) = H/K \rtimes TH/C_G(H/K) \in \mathfrak{F}.$$

Assume now that $H/K \rtimes TH/C_G(H/K) \in \mathfrak{F}$ for any $\mathfrak{H}$-subgroup $T/C_G(H/K)$ of $G/C_G(H/K)$. Let $L/K$ be an $\mathfrak{H}$-subgroup of $G/K$. Then

$$LC_G(H/K)/C_G(H/K) \simeq (L/K)C_{G/K}(H/K)/C_{G/K}(H/K) \in \mathfrak{H}.$$

So $H/K \rtimes LHC_G(H/K)/C_G(H/K) \in \mathfrak{F}$. Then

$$H/K \rtimes LHC_G(H/K)/C_G(H/K) \simeq H/K \rtimes LH/C_{LH}(H/K) \in \mathfrak{F}.$$

Let $H_1/K_1$ be the chief factor of $LH/K$ below $H/K$. As $\mathfrak{F}$ is a hereditary formation, then $H_1/K \rtimes LH/C_{LH}(H/K) \in \mathfrak{F}$. Since $K_1/K \trianglelefteq H_1/K \rtimes LH/C_{LH}(H/K)$, so $H_1/K_1 \rtimes LH/C_{LH}(H/K) \in \mathfrak{F}$. Note that $C_{LH}(H/K) \leq C_{LH}(H_1/K_1)$ and $C_{LH}(H_1/K_1)/C_{LH}(H/K) \trianglelefteq H_1/K_1 \rtimes LH/C_{LH}(H/K)$. So

$$H_1/K_1 \rtimes (LH/C_{LH}(H/K))/(C_{LH}(H_1/K_1)/C_{LH}(H/K)) \simeq$$
$$H_1/K_1 \rtimes LH/C_{LH}(H_1/K_1) \in \mathfrak{F}.$$

Thus all chief factors of $LH/K$ below $H/K$ are $\mathfrak{F}$-central. From

$$(LH/K)/(H/K) \simeq LH/H \simeq L/(L \cap H) \in \mathfrak{H} \subseteq \mathfrak{F}$$

it follows that $LH/K \in \mathfrak{F}$ by Lemma 8. So $L/K$ $\mathfrak{F}$-sn $LH/K$ for any $\mathfrak{H}$-subgroup $L/K$ of $G/K$. Hence $H/K \leq \mathcal{I}_{\mathfrak{H}}^{\mathfrak{F}}(G/K) = \mathrm{Int}_{\mathfrak{F}}(G/K)$ by Theorems 7 and 8. □

*Proof of the Corollary 3.* Let $\mathfrak{F}$ be a non-empty hereditary saturated formation and $f$ be its local definition (it exists by Gaschütz – Lubeseder – Schmid theorem). Now $\mathfrak{F} = f_{\mathfrak{F}}(\mathfrak{F})$ by Lemma 9. From Theorem 5 it follows that we need only to prove that $H/K \rtimes TH/C_G(H/K) \in \mathfrak{F}$ iff $TH/C_G(H/K) \in \mathfrak{N}_p f(p)$ for every $p \in \pi(H/K)$ where $H/K$ is a chief factor of $G$ and $T/C_G(H/K)$ is an $\mathfrak{F}$-subgroup of $G/C_G(H/K)$. Let $C = C_G(H/K)$ and $\overline{T} = (H/K) \rtimes (HT/C)$.

Assume that $TH/C_G(H/K) \in \mathfrak{N}_p f(p)$ for all $p \in \pi(H/K)$. Suppose that $H/K$ is non-abelian. Since $\mathfrak{N}_p f(p)$ is a formation, $(TH/C)/C_{TH/C}(\overline{U}) \in \mathfrak{N}_p f(p)$ for every chief factor $\overline{U}$ of $TH/C$ below $HC/C$ and every $p \in \pi(\overline{U})$. From

$$(TH/C)/(HC/C) \simeq TH/HC \simeq T/(T \cap HC) = T/C(T \cap H) \in \mathfrak{F}$$

and the definition of a local formation it follows that $TH/C \in \mathfrak{F}$.

Let $H^* = \{(h^{-1}K, hC) \mid h \in H\}$. It is an easy calculation to show that $H^* \trianglelefteq (H/K) \rtimes HT/C$, $H^* \cap (HT/C) \simeq 1$ and $H^*HT/C = (H/K) \rtimes (HT/C) = \overline{T}$. Now $\overline{T}/(H/K) \simeq \overline{T}/H^* \simeq HT/C \in \mathfrak{F}$. Hence $\overline{T} \in \mathfrak{F}$.

Assume that $H/K$ is abelian. Then $H/K$ is a $p$-group for some prime $p$ and $H \leq C$. So $HT/C = T/C \in \mathfrak{N}_p f(p) \cap \mathfrak{F}$. From $H/K \leq C_{\overline{T}}(H/K)$ it follows that $\overline{T}/C_{\overline{T}}(\overline{U}) \in \mathfrak{N}_p f(p) \cap \mathfrak{F}$ for every chief factor $\overline{U}$ of $\overline{T}$ below $H/K$. It means that $\overline{T} \in \mathfrak{F}$.

Suppose now that $(H/K) \rtimes (HT/C) \in \mathfrak{F}$. Since $\mathfrak{F}$ is a local formation, $\overline{T}/C_{\overline{T}}(H/K) \in \mathfrak{N}_p f(p)$ for all $p \in \pi(H/K)$ by [31, Lemma 3.6]. Note that $\overline{T} \leq \overline{G} = (H/K) \rtimes G/C$ is a primitive group of type 1 or 3. If $H/K$ is abelian, then $H/K = C_{\overline{G}}(H/K)$. So $\overline{T}/C_{\overline{T}}(H/K) \simeq HT/C \in \mathfrak{N}_p f(p)$ for all $p \in \pi(H/K)$. If $H/K$ is non-abelian, then $G/C$ complements $C_{\overline{G}}(H/K) = \{(h^{-1}K, hC) \mid h \in H\}$. It means $C_{\overline{G}}(H/K) = C_{\overline{T}}(H/K)$ and $\overline{T}/C_{\overline{T}}(H/K) \simeq TH/C \in \mathfrak{N}_p f(p)$ for all $p \in \pi(H/K)$. □

## 5 The computation of $\mathrm{Int}_{\mathfrak{F}}(G)$

The papers [32, 33, 34, 35] are dedicated to the computational theory of formations. In particular the computation of $\mathrm{Int}_{\mathfrak{F}}(G)$ is considered only in [35] for some family of local formations. The aim of this section is to combine our previous results with the results from [35] to develop the algorithms of the computation of $\mathrm{Int}_{\mathfrak{F}}(G)$ for a wide family of formations $\mathfrak{F}$.

**Definition 3** ([34, Definition 2], [35, Definition 2]). *Let* f *be a function which assigns 0 or 1 to every group $G$ and its chief factor $H/K$ such that*

*(1)* $\mathrm{f}(H/K, G) = \mathrm{f}(M/N, G)$ *whenever $H/K$ and $M/N$ are $G$-isomorphic chief factors of $G$;*

*(2)* $\mathrm{f}(H/K, G) = \mathrm{f}((H/N)/(K/N), G/N)$ *for every $N \trianglelefteq G$ with $N \leq K$.*

*Such functions* f *are called chief factor functions. Let* $\mathrm{Z}(G, \mathrm{f})$ *be the greatest normal subgroup of $G$ such that* $\mathrm{f}(H/K, G) = 1$ *for every chief factor $H/K$ of $G$ below it.*

According to [35, Theorem 1] $\mathrm{Z}(G, \mathrm{f})$ is well defined. Note that if f checks if $H/K$ is $\mathfrak{F}$-central in $G$, then f is a chief factor function and $\mathrm{Z}(G, \mathrm{f})$ is the $\mathfrak{F}$-hypercenter of $G$.

**Theorem 9.** *Let $\mathfrak{H}$ be a saturated homomorph and $\mathfrak{F} = f_{\mathfrak{H}}(\mathfrak{F})$ be a hereditary formation with $\mathfrak{H} \subseteq \mathfrak{F}$. If* f *checks if $H/K \rtimes TH/C_G(H/K) \in \mathfrak{F}$ for every $\mathfrak{H}$-subgroup $T/C_G(H/K)$ of $G/C_G(H/K)$ for a chief factor $H/K$ of a group $G$, then* f *is a chief factor function and* $\mathrm{Z}(G, \mathrm{f}) = \mathrm{Int}_{\mathfrak{F}}(G)$ *for every group $G$.*

*Proof.* (*a*) f *is a chief factor function*

Let $D = M \rtimes A$ and $R = N \rtimes B$. Recall that the pairs $(M, A)$ and $(N, B)$ are said to be equivalent [4, 3.26] provided there are isomorphisms $f : M \to N$ and $g : A \to B$ such that $f(a^{-1}ma) = g(a^{-1})f(m)g(a)$ for all $m \in M$ and $a \in A$. If $H/K$ and $M/N$ are $G$-isomorphic chief factors of $G$, then $C_G(H/K) = C_G(M/N)$ and it is straightforward to check that $(H/K, G/C_G(H/K))$ and $(M/N, G/C_G(M/N))$ are equivalent pairs. According to [4, Lemma 3.27] the map $\varphi$ defined by $\varphi(ma) = f(m)g(a)$, where $a \in A$ and $m \in M$, is the isomorphism between $D$ and $R$. Note that if $H/K$ is abelian, then $HM \subseteq C_G(H/K) = C_G(M/N)$. If $H/K$ is non-abelian, then $HC_G(H/K)/C_G(H/K) = MC_G(M/N)/C_G(M/N)$ is the unique minimal normal subgroup of $G/C_G(H/K)$. It means that

$$H/K \rtimes TH/C_G(H/K) \simeq M/N \rtimes MT/C_M(M/N)$$

for any $\mathfrak{H}$-subgroup $T/C_G(H/K) = T/C_M(M/N)$. Therefore f satisfies (1) of Definition 3.

Let $H/K$ be a chief factor of $G$ and $N \trianglelefteq G$ with $N \leq K$. From $C_{G/N}((H/N)/(K/N)) = C_G(H/K)/N$ it follows that $T/C_G(H/K)$ is an $\mathfrak{H}$-subgroup of $G/C_G(H/K)$ iff $(T/N)/C_{G/N}((H/N)/(K/N))$ is an $\mathfrak{H}$-subgroup of

$(G/N)/C_{G/N}((H/N)/(K/N))$. Since

$$H/K \rtimes TH/C_G(H/K) \simeq (H/N)/(K/N) \rtimes (H/N)(T/N)/C_{G/N}((H/N)/(K/N))$$

for every subgroup $T/C_G(H/K)$ of $G/C_G(H/K)$, we see that f satisfies (2) of Definition 3. Thus f is a chief factor function.

($b$) $\mathrm{Z}(G, \mathrm{f}) = \mathrm{Int}_{\mathfrak{F}}(G)$.

Directly follows from (1) of Theorem 4, Theorem 5 and Definition 3. □

The computations in permutation groups are rather developed nowadays. We refer the reader to the book [36] for the basic results in this field. The polynomial time algorithm to compute $\mathrm{Z}(G/K, \mathrm{f})$ was suggested in [35].

**Theorem 10.** *Let $\mathfrak{F} = w\mathfrak{F}$ be a hereditary formation. Assume that $G^{\mathfrak{F}}$ can be computed in a polynomial time in $n$ for every $G \leq S_n$. Then $\mathrm{Int}_{\mathfrak{F}}(G/K)$ can be computed in a polynomial time in $n$ for every $K \trianglelefteq G \leq S_n$.*

*Proof.* Let $\pi = \pi(\mathfrak{F})$ and $\mathfrak{H}$ be a class of all primary $\pi$-groups. Then $\mathfrak{F} = f_{\mathfrak{H}}(\mathfrak{F})$.

Let prove for a chief factor $H/K$ of $G$ that $H/K \rtimes TH/C_G(H/K) \in \mathfrak{F}$ for every $\mathfrak{H}$-subgroup $T/C_G(H/K)$ of $G/C_G(H/K)$ iff $HP/K \in \mathfrak{F}$ for every Sylow subgroup $P$ of $G$.

Assume that $H/K \rtimes TH/C_G(H/K) \in \mathfrak{F}$ for every $\mathfrak{H}$-subgroup $T/C_G(H/K)$ of $G/C_G(H/K)$. Let $P$ be a Sylow $\pi$-subgroup of $G$. Since $PC_G(H/K)/C_G(H/K) \in \mathfrak{H}$, we see that $H/K \rtimes PH/C_{PH}(H/K) \simeq H/K \rtimes HPC_G(H/K)/C_G(H/K) \in \mathfrak{F}$. From [27, Lemma 2.2] it follows that all chief factors of $PH/K$ below $H/K$ are $\mathfrak{F}$-central. Since $P$ is a primary $\pi$-group, all chief factors of $PH/K$ above $H/K$ are $\mathfrak{F}$-central. Thus $PH/K \in \mathfrak{F}$ by Lemma 8.

Assume that $HP/K \in \mathfrak{F}$ for every Sylow $\pi$-subgroup $P$ of $G$. Then $H/K \rtimes HP/C_{HP}(H/K) \simeq H/K \rtimes HPC_G(H/K)/C_G(H/K) \in \mathfrak{F}$ by Theorem 6 for every Sylow $\pi$-subgroup $P$ of $G$. If $T/C_G(H/K)$ is a primary $\pi$-subgroup of $G/C_G(H/K)$, then there exists a Sylow $\pi$-subgroup $P$ of $G$ with $T/C_G(H/K) \leq PC_G(H/K)/C_G(H/K)$. Hence $H/K \rtimes HT/C_G(H/K) \in \mathfrak{F}$ as a subgroup of an $\mathfrak{F}$-group.

From Theorem 9 it follows that $\mathrm{Z}(G, \mathrm{f}) = \mathrm{Int}_{\mathfrak{F}}(G)$ holds for every group $G$ where a chief factor function f checks if $HP/K \in \mathfrak{F}$ for every Sylow $(\pi \cap \pi(G))$-subgroup $P$ of $G$ (including 1 in case $\pi \cap \pi(G) = \emptyset$).

To do this check we need only to find exactly one Sylow $p$-subgroup for every $p \in \pi \cap \pi(G)$. This can be done by W. Kantor's theorem [37]. From [34, Theorem 11] it follows that we can check if $HP/K \in \mathfrak{F}$ in polynomial time. Therefore the values of f can be computed in polynomial time for every chief factor $H/K$ of $G$. Thus $\mathrm{Z}(G/K, \mathrm{f}) = \mathrm{Int}_{\mathfrak{F}}(G/K)$ can be computed in polynomial time for every $K \trianglelefteq G \leq S_n$ by [35, Theorem 1]. □

The class $w\mathfrak{U}$ of widely supersoluble groups was introduced in [38]. It has a lot of interesting applications nowadays (for example, see [39]).

**Corollary 5.** $\mathrm{Int}_{w\mathfrak{U}}(G/K)$ *can be computed in a polynomial time in $n$ for every $K \trianglelefteq G \leq S_n$.*

A. Lucchini and D. Nemmi introduced the concept of a regular formation and studied such formations of soluble groups. Recall [12] that a hereditary formation $\mathfrak{F}$ is called regular if in every group $G$ holds $\mathrm{Int}_{\mathfrak{F}}(G) = \{x \mid \langle x, y\rangle \in \mathfrak{F} \text{ for every } y \in G\}$. Such formations $\mathfrak{F}$ play an important role in the study of $\mathfrak{F}$-graphs and

non-$\mathfrak{F}$-graphs of a group because the set of vertices of these graphs is $G \setminus \mathrm{Int}_{\mathfrak{F}}(G)$ in this case.

**Theorem 11.** *Let $(1) \subset \mathfrak{F}$ be a regular saturated formation of soluble groups and $F$ be its canonical local definition. If* f *checks if $H/K$ is abelian and $exp(G/C_G(H/K))$ divides $exp(F(p))$ where $\{p\} = \pi(H/K)$ for a chief factor $H/K$ of $G$, then* f *is a chief factor function and $\mathrm{Z}(G, \mathrm{f}) = \mathrm{Int}_{\mathfrak{F}}(G)$ for every group $G$.*

*Proof.* Note that $\pi(\mathfrak{F}) = \mathbb{P}$. Let $\mathfrak{H}$ be a class of all cyclic primary groups. Since $\mathfrak{F}$ is saturated, $\mathfrak{H} \subseteq \mathfrak{F}$. From [19, Theorem B] and [40, Theorem 1] it follows that $\mathfrak{F} = v\mathfrak{F}$, i.e. $\mathfrak{F} = f_{\mathfrak{H}}(\mathfrak{F})$. Since $F$ is the canonical local definition of $\mathfrak{F}$, we see that $v_p(exp(F(p))) = \infty$ for every $p \in \mathbb{P}$.

Let prove for a chief factor $H/K$ of $G$ that $H/K \rtimes TH/C_G(H/K) \in \mathfrak{F}$ for every $\mathfrak{H}$-subgroup $T/C_G(H/K)$ of $G/C_G(H/K)$ iff $H/K$ is abelian and $exp(G/C_G(H/K))$ divides $exp(F(p))$ where $\{p\} = \pi(H/K)$.

Assume that $H/K \rtimes TH/C_G(H/K) \in \mathfrak{F}$ for every $\mathfrak{H}$-subgroup $T/C_G(H/K)$ of $G/C_G(H/K)$. Since $\mathfrak{F}$ is soluble, $H/K$ is abelian. Hence $H/K \leq C_G(H/K)$. Note that $exp(G/C_G(H/K)) = \mathrm{lcm}\{exp(T/C_G(H/K)) \mid T/C_G(H/K) \in \mathfrak{H}\}$. Let $\overline{T} = T/C_G(H/K) \in \mathfrak{H}$. If $\overline{T}$ is a $p$-group, then $exp(\overline{T})$ divides $exp(F(p))$. Assume that $\overline{T}$ is a $q$-group for some prime $q \neq p$. Let $M = H/K \rtimes \overline{T} \in \mathfrak{F}$. From $C_{\overline{T}}(H/K) \simeq 1$ it follows that $\mathrm{O}_{p',p}(M) = H/K$. Hence $\overline{T} \simeq M/C_M(H/K) \in F(p)$. So $exp(\overline{T})$ divides $exp(F(p))$. Thus $exp(G/C_G(H/K))$ divides $exp(F(p))$.

Assume that $H/K$ is abelian and $exp(G/C_G(H/K))$ divides $exp(F(p))$ where $\{p\} = \pi(H/K)$. Note that $H/K \leq C_G(H/K)$. Let $\overline{T} = T/C_G(H/K) \in \mathfrak{H}$ and $M = H/K \rtimes HT/C_G(H/K) = H/K \rtimes \overline{T}$. If $\overline{T}$ is a $p$-group, then $M \in \mathfrak{N}_p \subseteq \mathfrak{F}$. Assume that $\overline{T}$ is a $q$-group for some prime $q \neq p$. Since $\overline{T} \in \mathfrak{H}$ and $exp(G/C_G(H/K))$ divides $exp(F(p))$, we see that $\overline{T} \in F(p)$. Now $M/C_M(A/B) \in F(r)$ for every chief factor $A/B$ of $M$ and $r \in \pi(A/B)$. So $M \in \mathfrak{F}$. Thus $H/K \rtimes TH/C_G(H/K) \in \mathfrak{F}$ for every $\mathfrak{H}$-subgroup $T/C_G(H/K)$ of $G/C_G(H/K)$.

Now Theorem 11 directly follows from Theorem 9. □

## References


[1] R. Baer, *Supersoluble immersion*, Can. J. Math., **11** (1959), 353–369. Zbl 0088.02402

[2] B. Huppert, *Zur theorie der formationen*, Arch. Math., **19**:6 (1969), 561–574. Zbl 0192.35303

[3] L.A. Shemetkov, *Graduated formations of groups*, Math. USSR-Sb., **23**:4 (1974), 593–611. Zbl 0323.20036

[4] L.A. Shemetkov, A.N. Skiba, *Formations of algebraic systems*, Nauka, Moskow, 1989. Zbl 0667.08001

[5] W. Guo, *Structure theory for canonical classes of finite groups*, Springer, Berlin, 2015. Zbl 1343.20021

[6] R. Baer, *Group theoretical properties and functions*, Colloq. Math., **14** (1966), 285–327. Zbl 0143.03804

[7] R. Baer, *Group elements of prime power index*, Trans. Am. Math. Soc., **75**:1 (1953), 20–47. Zbl 0051.25702

[8] A.N. Skiba, *On the intersection of all maximal $\mathfrak{F}$-subgroups of a finite group*, J. Algebra, **343**:1 (2011), 173–182. Zbl 1247.20020

[9] A.N. Skiba, *On the $\mathfrak{F}$-hypercentre and the intersection of all $\mathfrak{F}$-maximal subgroups of a finite group*, J. Pure Appl. Algebra, **216**:4 (2012), 789–799. Zbl 1261.20015
[10] W. Guo, A.N. Skiba, *On the intersection of the $\mathfrak{F}$-maximal subgroups and the generalized $\mathfrak{F}$-hypercentre of a finite group*, J. Algebra, **366** (2012), 112–125. Zbl 1281.20021
[11] J.C. Beidleman, H. Heineken, *A note of intersection of maximal $\mathfrak{F}$-subgroups*, J. Algebra , **333**:1 (2011), 120–127. Zbl 1269.20012
[12] A. Lucchini, D. Nemmi, *The non-$\mathfrak{F}$ graph of a finite group*, Math. Nachr., **294**:10 (2021), 1912–1921. Zbl 1522.05192
[13] V.I. Murashka, A.F. Vasil'ev, *On the $\sigma$-nilpotent hypercenter of finite groups*, J. Group Theory, **25**:6 (2022), 1083–1098. Zbl 1509.20022
[14] A. Ballester-Bolinches, L.M. Ezquerro, *Classes of finite groups*, Springer, Dordrecht, 2006. Zbl 1102.20016
[15] K. Doerk, T. Hawkes, *Finite soluble groups*, W. de Gruyter, Berlin etc., 1992. Zbl 0753.20001
[16] L.A. Shemetkov, *Frattini extensions of finite groups and formations*, Commun. Algebra, **25**:3 (1997), 955–964. Zbl 0886.20010
[17] V.I. Murashka, *On questions posed by Shemetkov, Ballester-Bolinches, and Perez-Ramos in finite group theory*, Math. Notes, **112**:6 (2022), 932–939. Zbl 1555.20025
[18] V.I. Murashka, *Finite groups with given sets of $\mathfrak{F}$-subnormal subgroups*, Asian-Eur. J. Math., **13**:4 (2020), Article ID 2050073. Zbl 1454.20028
[19] V.I. Murashka, *Classes of finite groups with generalized subnormal cyclic primary subgroups*, Sib. Math. J., **55**:6 (2014), 1105–1115. Zbl 1344.20028
[20] A.F. Vasil'ev, T.I. Vasilyeva, *On finte groups with generally subnormal Sylow subgroups*, Probl. Fiz. Mat. Tekh., **2011**:4(9) (2011), 86–91. Zbl 1245.20021
[21] A.F. Vasil'ev, T.I. Vasil'eva, A.S. Vegera, *Finite groups with generalized subnormal embedding of Sylow subgroups*, Sib. Math. J., **57**:2 (2016), 200–212. Zbl 1393.20006
[22] V.N. Semenchuk, S.N. Shevchuk, *Characterization of classes of finite groups withthe use of generalized subnormal Sylow subgroups*, Math. Notes., **89**:1 (2011), 117–120. Zbl 1230.20018
[23] V.S. Monakhov, I.L. Sokhor, *On groups with formational subnormal Sylow subgroups*, J. Group Theory, **21**:2 (2018), 273–287. Zbl 1468.20041
[24] W. Guo, A.N. Skiba, *Finite groups whose n-maximal subgroups are $\sigma$-subnormal*, Sci. China Math., **62**:7 (2019), 1355–1372. Zbl 1476.20020
[25] A. Ballester-Bolinches, S.F. Kamornikov, X. Yi, *Finite groups with $\sigma$-subnormal Schmidt subgroups*, Bull. Malays. Math. Sci. Soc. (2), **45**:5 (2022), 2431–2440. Zbl 1514.20060
[26] S.F. Kamornikov, M.V. Selkin, *Subgroup functors and classes of finite groups*, Belaruskaya Nauka, Minsk, 2003.
[27] S. Aivazidis, I.N. Safonova, A.N. Skiba, *Subnormality and residuals for saturated formations: a generalization of Schenkman's theorem*, J. Group Theory, **24**:4 (2021), 807–818. Zbl 1482.20012
[28] R.W. Carter, *Nilpotent self-normalizing subgroups and system normalizers*, Proc. London Math. Soc., III. Ser., **12**:1 (1962), 535–563. Zbl 0106.24602

[29] C.J. Graddon, *The relation between $\mathfrak{F}$-reducers and $\mathfrak{F}$-subnormalizers in finite soluble groups*, J. Lond. Math. Soc., II. Ser., **4**:1 (1971), 51–61. Zbl 0221.20021
[30] A. Mann, *System normalizers and subnormalizers*, Proc. Lond. Math. Soc., III. Ser., **20**:1 (1970), 123–143. Zbl 0214.04402
[31] V.I. Murashka, *Products of $F^*(G)$-subnormal subgroups of finite groups*, J. Algebra Appl., **22**:9 (2023), Article ID 2350184. Zbl 1521.20037
[32] B. Eick, C. Wright, *Computing subgroups by exhibition in finite solvable groups*, J. Symb. Comput., **33**:2 (2002), 129–143. Zbl 0995.20004
[33] B. Höfling, *Computing projectors, injectors, residuals and radicals of finite solvable groups*, J. Symb. Comput., **32**:5 (2001), 499–511. Zbl 0990.20008
[34] V.I. Murashka, *Formations of finite groups in polynomial time: $\mathfrak{F}$-residuals and $\mathfrak{F}$-subnormality*, J. Symb. Comput., **122** (2024), Article ID 102271. Zbl 1529.20020
[35] V.I. Murashka, *Formations of finite groups in polynomial time II: the $\mathfrak{F}$-hypercenter and its generalizations*, Tr. Inst. Mat. Mekh. UrO RAN, **31**:1 (2025), 154–165. MR4889594
[36] Á. Seress, *Permutation group algorithms*, Cambridge Tracts in Mathematics, **152**, Cambridge University Press, Cambridge, 2003. Zbl 1028.20002
[37] W.M. Kantor, *Sylow's theorem in polynomial time*, J. Comput. Syst. Sci., **30**:3 (1985), 359–394. Zbl 0573.20022
[38] A.F. Vasil'ev, T.I. Vasil'eva, V.N. Tyutyanov, *On the finite groups of supersolvable type*, Sib. Math. J., **51**:6 (2010), 1004–1012. Zbl 1226.20013
[39] A. Ballester-Bolinches, S.Y. Madanha, M.C. Pedraza-Aguilera, X. Wu, *On some products of finite groups*, Proc. Edinb. Math. Soc., II. Ser., **66**:1 (2023), 89–99. Zbl 1517.20030
[40] V.I. Murashka, *Regular saturated formations of finite soluble groups*, J. Group Theory., **28**:6 (2025), 1307–1324. Zbl 1581.20013

Viacaslau Igaravich Murashka, Yana Aleksandrovna Kuptsova
Francisk Skorina Gomel State University,
str. Sovetskaya, 104,
246028, Gomel, Belarus
*Email address*: mvimath@yandex.by, kuptsovayana519@gmail.com