

**ALGEBRAS OF BINARY FORMULAS FOR
CIRCULARLY ORDERED STRUCTURES: PIECEWISE
MONOTONIC CASE****B.SH. KULPESHOV** *Communicated by S.V. SUDOPLATOV*

Abstract: This article concerns the notion of weak circular minimality being a variant of o-minimality for circularly ordered structures. Algebras of binary isolating formulas are studied for $\aleph_0$ -categorical 1-transitive non-primitive weakly circularly minimal theories of convexity rank greater than 1 with a trivial definable closure having a piecewise monotonic-to-left function to the definable completion of a structure. On the basis of the study, we present a description of these algebras. It is shown that for this case there exist only non-commutative algebras. A strict s -deterministicity of such algebras for some natural number s is also established.

Keywords: algebra of binary formulas, weak circular minimality, $\aleph_0$ -categorical theory, circularly ordered structure, convexity rank.

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1 Preliminaries

Algebras of binary formulas are a tool for describing relationships between elements of the sets of realizations of an one-type at the binary level with respect to the superposition of binary definable sets. A *binary isolating formula* is a formula of the form $\varphi(x, y)$ such that for some parameter a the formula $\varphi(a, y)$ isolates a complete type in $S(\{a\})$. The concepts and notations related to these algebras can be found in the papers [22, 23]. In recent years, algebras of binary formulas have been studied intensively and have been continued in the works [1], [6]–[15].

Let L be a countable first-order language. Throughout we consider L -structures and assume that L contains a ternary relational symbol K , interpreted as a circular order in these structures (unless otherwise stated).

The *circular order* is described by a ternary relation K satisfying the following conditions:

- (co1) $\forall x \forall y \forall z (K(x, y, z) \rightarrow K(y, z, x))$;
- (co2) $\forall x \forall y \forall z (K(x, y, z) \wedge K(y, x, z) \Leftrightarrow x = y \vee y = z \vee z = x)$;
- (co3) $\forall x \forall y \forall z (K(x, y, z) \rightarrow \forall t [K(x, y, t) \vee K(t, y, z)])$;
- (co4) $\forall x \forall y \forall z (K(x, y, z) \vee K(y, x, z))$.

The following observation relates linear and circular orders.

Fact 1. [3] (i) If $\langle M, \leq \rangle$ is a linear ordering and K is the ternary relation derived from $\leq$ by the rule

$$K(x, y, z) :\Leftrightarrow (x \leq y \leq z) \vee (z \leq x \leq y) \vee (y \leq z \leq x)$$

then K is a circular order relation on M .

(ii) If $\langle N, K \rangle$ is a circular ordering and $a \in N$, then the relation $\leq_a$ defined on $M := N \setminus \{a\}$ by the rule $y \leq_a z :\Leftrightarrow K(a, y, z)$ is a linear order.

Thus, any linearly ordered structure is circularly ordered, since the relation of circular order is $\emptyset$ -definable in an arbitrary linearly ordered structure. However, the opposite is not true. The following example shows that there are circularly ordered structures not being linearly ordered.

Example 1. [4, 5] Let $\mathbb{Q}_2^* := \langle \mathbb{Q}_2, K, L \rangle$ be a circularly ordered structure, where $L = \{\sigma_0^2, \sigma_1^2\}$, for which the following conditions hold:

- (i) its domain $\mathbb{Q}_2$ is a countable dense subset of the unit circle, no two points making the central angle π ;
- (ii) for distinct $a, b \in \mathbb{Q}_2$

$$(a, b) \in \sigma_0 \Leftrightarrow 0 < \arg(a/b) < \pi, \quad (a, b) \in \sigma_1 \Leftrightarrow \pi < \arg(a/b) < 2\pi,$$

where $\arg(a/b)$ means the value of the central angle between a and b clockwise. Indeed, one can check that the linear order relation is not $\emptyset$ -definable in this structure.

The notion of *weak circular minimality* was studied initially in [16]. Let $A \subseteq M$, where M is a circularly ordered structure. The set A is called *convex* if for any $a, b \in A$ the following property is satisfied: for any $c \in M$

with $K(a, c, b)$, $c \in A$ holds, or for any $c \in M$ with $K(b, c, a)$, $c \in A$ holds. A *weakly circularly minimal structure* is a circularly ordered structure $M = \langle M, K, \dots \rangle$ such that any definable (with parameters) subset of M is a union of finitely many convex sets in M . The study of weakly circularly minimal structures was continued in the papers [2], [16]–[18].

Let M be an $\aleph_0$ -categorical weakly circularly minimal structure, $G := \text{Aut}(M)$. Following the standard group theory terminology, the group G is called *k-transitive* if for any pairwise distinct $a_1, a_2, \dots, a_k \in M$ and pairwise distinct $b_1, b_2, \dots, b_k \in M$ there exists $g \in G$ such that $g(a_1) = b_1, g(a_2) = b_2, \dots, g(a_k) = b_k$. A *congruence* on M is an arbitrary G -invariant equivalence relation on M . The group G is called *primitive* if G is 1-transitive and there are no non-trivial proper congruences on M .

Notation 1. (1) $K_0(x, y, z) := K(x, y, z) \wedge y \neq x \wedge y \neq z \wedge x \neq z$.

(2) $K(u_1, \dots, u_n)$ denotes a formula saying that all sub tuples of the tuple $\langle u_1, \dots, u_n \rangle$ having the length 3 (in ascending order) satisfy K ; similar notations are used for K_0 .

(3) Let A, B, C be disjoint convex subsets of a circularly ordered structure M . We write $K(A, B, C)$ if for any $a, b, c \in M$ with $a \in A, b \in B, c \in C$ we have $K(a, b, c)$. We extend naturally that notation using, for instance, the notation $K_0(A, d, B, C)$ if $d \notin A \cup B \cup C$ and $K_0(A, d, B) \wedge K_0(d, B, C)$ holds.

Further we need the notion of the definable completion of a circularly ordered structure, introduced in [16]. Its linear analog was introduced in [21]. A *cut* $C(x)$ in a circularly ordered structure M is maximal consistent set of formulas of the form $K(a, x, b)$, where $a, b \in M$. A cut is said to be *algebraic* if there exists $c \in M$ that realizes it. Otherwise, such a cut is said to be *non-algebraic*. Let $C(x)$ be a non-algebraic cut. If there is some $a \in M$ such that either for all $b \in M$ the formula $K(a, x, b) \in C(x)$, or for all $b \in M$ the formula $K(b, x, a) \in C(x)$, then $C(x)$ is said to be *rational*. Otherwise, such a cut is said to be *irrational*. A *definable cut* in M is a cut $C(x)$ with the following property: there exist $a, b \in M$ such that $K(a, x, b) \in C(x)$ and the set $\{c \in M \mid K(a, c, b) \text{ and } K(a, x, c) \in C(x)\}$ is definable. The *definable completion* $\bar{M}$ of a structure M consists of M together with all definable cuts in M that are irrational (essentially $\bar{M}$ consists of endpoints of definable subsets of the structure M).

Notation 2. [16] Let $F(x, y)$ be an L -formula such that $F(M, b)$ is convex infinite co-infinite for each $b \in M$. Let $F^\ell(y)$ be the formula saying y is a left endpoint of $F(M, y)$:

$$\begin{aligned} & \exists z_1 \exists z_2 [K_0(z_1, y, z_2) \wedge \forall t_1 (K(z_1, t_1, y) \wedge t_1 \neq y \rightarrow \neg F(t_1, y)) \wedge \\ & \quad \forall t_2 (K(y, t_2, z_2) \wedge t_2 \neq y \rightarrow F(t_2, y))]. \end{aligned}$$

We say that $F(x, y)$ is *convex-to-right* if

$$M \models \forall y \forall x [F(x, y) \rightarrow F^\ell(y) \wedge \forall z (K(y, z, x) \rightarrow F(z, y))].$$

If $F_1(x, y), F_2(x, y)$ are arbitrary convex-to-right formulas we say F_2 is *bigger than* F_1 if there is $a \in M$ with $F_1(M, a) \subset F_2(M, a)$. If M is 1-transitive and this holds for some a , it holds for all a . This gives a total ordering on the (finite) set of all convex-to-right formulas $F(x, y)$ (viewed up to equivalence modulo $Th(M)$).

Consider $F(M, a)$ for arbitrary $a \in M$. In general, $F(M, a)$ has no the right endpoint in M . For example, if $dcl(\{a\}) = \{a\}$ holds for some $a \in M$ then for any convex-to-right formula $F(x, y)$ and any $a \in M$ the formula $F(M, a)$ has no the right endpoint in M . We write $f(y) := \text{rend } F(M, y)$, assuming that $f(y)$ is the right endpoint of the set $F(M, y)$ that lies in general in the definable completion $\overline{M}$ of M . Then f is a function mapping M in $\overline{M}$.

Notation 3. Let $E(x, y)$ be an $\emptyset$ -definable equivalence relation partitioning M into infinite convex classes. Suppose that y lies in $\overline{M}$ (non-obligatory in M). Then

$$E^*(x, y) := \exists y_1 \exists y_2 [y_1 \neq y_2 \wedge \forall t (K(y_1, t, y_2) \rightarrow E(t, x)) \wedge K_0(y_1, y, y_2)].$$

Let M, N be circularly ordered structures. The *2-reduct* of M is a circularly ordered structure with the same universe of M and consisting of predicates for each $\emptyset$ -definable relation on M of arity ≤ 2 as well as of the ternary predicate K for the circular order, but does not have other predicates of arities more than two. We say that the structure M is *isomorphic* to N *up to binarity* or *binarily isomorphic* to N if the 2-reduct of M is isomorphic to the 2-reduct of N .

Let f be a unary function from M to $\overline{M}$. We say that f is *monotonic-to-right (left) on* M if it preserves (reverses) the relation K_0 , i.e. for any $a, b, c \in M$ such that $K_0(a, b, c)$, we have $K_0(f(a), f(b), f(c))$ ($K_0(f(c), f(b), f(a))$).

We also say that f is *piecewise monotonic-to-right (left) on* M if there exists an $\emptyset$ -definable non-trivial equivalence relation $E(x, y)$ partitioning M into finitely many infinite convex classes so that f is monotonic-to-right on each E -class and f is not monotonic-to-left (right) on M/E , where by M/E we denote the set of representatives of E -classes in M .

Example 2. [17] Let $M := \langle M, K, E^2, f^1 \rangle$ be a circularly ordered structure, where M is a disjoint union of $\mathbb{Q}_1, \mathbb{Q}_2, \dots, \mathbb{Q}_6$, where $\mathbb{Q}_i$ is a copy of the ordering of rational numbers $\mathbb{Q}$. The symbol E interprets an equivalence relation on M as follows: $E(a, b)$ iff there is $1 \leq i \leq 6$ with $a, b \in \mathbb{Q}_i$. The symbol f interprets a function on M as follows: $f(\mathbb{Q}_i) = \mathbb{Q}_{i+3}$ for each $1 \leq i \leq 3$, $f(\mathbb{Q}_j) = \mathbb{Q}_{j-3}$ for each $4 \leq j \leq 6$, and $f(q) = -q$ for all $q \in \mathbb{Q}$.

It can be proved that M is an $\aleph_0$ -categorical 1-transitive weakly circularly minimal structure, f is a bijection on M so that $f^2(a) = a$ for all $a \in M$, f is monotonic-to-left on each E -class and f is monotonic-to-right on M/E , i.e. f is piecewise monotonic-to-left on M .

The following definition can be used in a circular ordered structure as well.

Definition 1. [19], [20] Let T be a weakly o-minimal theory, M be a sufficiently saturated model of T , $A \subseteq M$. The rank of convexity of the set A ($RC(A)$) is defined as follows:

- 1) $RC(A) = -1$ if $A = \emptyset$.
- 2) $RC(A) = 0$ if A is finite and non-empty.
- 3) $RC(A) \geq 1$ if A is infinite.
- 4) $RC(A) \geq \alpha + 1$ if there exist a parametrically definable equivalence relation $E(x, y)$ and an infinite sequence of elements $b_i \in A, i \in \omega$, such that:

- For every $i, j \in \omega$ whenever $i \neq j$ we have $M \models \neg E(b_i, b_j)$;
- For every $i \in \omega$, $RC(E(x, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of A .

- 5) $RC(A) \geq \delta$ if $RC(A) \geq \alpha$ for all $\alpha < \delta$, where δ is a limit ordinal.

If $RC(A) = \alpha$ for some α , we say that $RC(A)$ is defined. Otherwise (i.e. if $RC(A) \geq \alpha$ for all α), we put $RC(A) = \infty$.

The rank of convexity of a formula $\phi(x, \bar{a})$, where $\bar{a} \in M$, is defined as the rank of convexity of the set $\phi(M, \bar{a})$, i.e. $RC(\phi(x, \bar{a})) := RC(\phi(M, \bar{a}))$.

The rank of convexity of an 1-type p is defined as the rank of convexity of the set $p(M)$, i.e. $RC(p) := RC(p(M))$.

In particular, a theory has convexity rank 1 if there is no definable (with parameters) equivalence relations with infinitely many infinite convex classes.

The following theorem characterizes up to binarity $\aleph_0$ -categorical 1-transitive non-primitive weakly circularly minimal structures M of convexity rank greater than 1 having both a trivial definable closure and a convex-to-right formula $R(x, y)$ such that $r(y) := \text{rend } R(M, y)$ is piecewise monotonic-to-left on M :

Theorem 1. [18] Let M be an $\aleph_0$ -categorical 1-transitive non-primitive weakly circularly minimal structure of convexity rank greater than 1, $\text{dcl}(\{a\}) = \{a\}$ for some $a \in M$. Suppose that there exists a convex-to-right formula $R(x, y)$ such that $r(y) := \text{rend } R(M, y)$ is piecewise monotonic-to-left on M . Then M is isomorphic up to binarity to

$$M'_{s,m,k} := \langle M, K^3, E_1^2, E_2^2, \dots, E_s^2, E_{s+1}^2, R^2 \rangle,$$

where M is a circularly ordered structure, M is densely ordered, $s \geq 1$; E_{s+1} is an equivalence relation partitioning M into m infinite convex classes without endpoints; E_i for every $1 \leq i \leq s$ is an equivalence relation partitioning every E_{i+1} -class into infinitely many infinite convex E_i -subclasses without endpoints so that the induced order on E_i -subclasses is dense without endpoints; $R(M, a)$ has no right endpoint in M and $r^k(a) = a$ for all $a \in M$ and some $k \geq 2$, where $r^k(y) := r(r^{k-1}(y))$; for every $1 \leq i \leq s+1$ and any $a \in M$

$$M'_{s,m,k} \models \neg E_i^*(a, r(a)) \wedge \forall y (E_i(y, a) \rightarrow \exists u [E_i^*(u, r(a)) \wedge E_i^*(u, r(y))]),$$

$m \geq 4$, k is even and k divides m ; r is monotonic-to-left on every E_{s+1} -class and r is monotonic-to-right on M/E_{s+1} .

In [12] algebras of binary isolating formulas are described for $\aleph_0$ -categorical weakly circularly minimal theories of convexity rank 1 with a 1-transitive non-primitive automorphism group and a non-trivial definable closure. In [13]–[14] algebras of binary isolating formulas are described for $\aleph_0$ -categorical weakly circularly minimal theories of convexity rank greater than 1 with a 1-transitive non-primitive automorphism group and a non-trivial definable closure. In [15] algebras of binary isolating formulas are described for $\aleph_0$ -categorical weakly circularly minimal theories of convexity rank 1 with a 1-transitive non-primitive automorphism group and a trivial definable closure. Here we describe algebras of binary isolating formulas for $\aleph_0$ -categorical weakly circularly minimal theories of convexity rank greater than 1 with a 1-transitive non-primitive automorphism group and a trivial definable closure having a piecewise monotonic-to-left function to the definable completion of a structure for arbitrary s and $m = k = 4$.

2 Results

Let M be an 1-transitive structure. We denote every binary isolating formula acting in M by a label $u \in \rho_M$, where ρ_M denotes the set of all labels for the algebra $\mathcal{P}_M$ of binary isolating formulas of the structure M .

Definition 2. [23] The algebra $\mathcal{P}_M$ is said to be *deterministic* if $u_1 \cdot u_2$ is a singleton for any labels $u_1, u_2 \in \rho_M$.

Generalizing the last definition, we say that the algebra $\mathcal{P}_M$ is *m-deterministic* if the product $u_1 \cdot u_2$ consists of at most m elements for any labels $u_1, u_2 \in \rho_M$. We also say that an *m-deterministic* algebra $\mathcal{P}_M$ is *strictly m-deterministic* if it is not $(m - 1)$ -deterministic.

Example 3. Consider the structure $M'_{1,4,4} := \langle M, K^3, E_1^2, E_2^2, R^2 \rangle$ from Theorem 1 with the condition that the function $r(y) := \text{rend } R(M, y)$ is piecewise monotonic-to-left on M .

We assert that $Th(M'_{1,4,4})$ has seventeen binary isolating formulas:

$$\begin{aligned}
 \theta_0(x, y) &:= x = y, \\
 \theta_1(x, y) &:= K_0(x, y, r(x)) \wedge E_1(x, y), \\
 \theta_2(x, y) &:= K_0(x, y, r(x)) \wedge E_2(x, y) \wedge \neg E_1(x, y), \\
 \theta_3(x, y) &:= K_0(x, y, r(x)) \wedge E_2^*(y, r(x)) \wedge \neg E_1^*(y, r(x)), \\
 \theta_4(x, y) &:= K_0(x, y, r(x)) \wedge E_1^*(y, r(x)), \\
 \theta_5(x, y) &:= K_0(r(x), y, r^2(x)) \wedge E_1^*(y, r(x)), \\
 \theta_6(x, y) &:= K_0(r(x), y, r^2(x)) \wedge E_2^*(y, r(x)) \wedge \neg E_1^*(y, r(x)), \\
 \theta_7(x, y) &:= K_0(r(x), y, r^2(x)) \wedge E_2^*(y, r^2(x)) \wedge \neg E_1^*(y, r^2(x)), \\
 \theta_8(x, y) &:= K_0(r(x), y, r^2(x)) \wedge E_1^*(y, r^2(x)),
 \end{aligned}$$

$$\begin{aligned}
\theta_9(x, y) &:= K_0(r^2(x), y, r^3(x)) \wedge E_1^*(y, r^2(x)), \\
\theta_{10}(x, y) &:= K_0(r^2(x), y, r^3(x)) \wedge E_2^*(y, r^2(x)) \wedge \neg E_1^*(y, r^2(x)), \\
\theta_{11}(x, y) &:= K_0(r^2(x), y, r^3(x)) \wedge E_2^*(y, r^3(x)) \wedge \neg E_1^*(y, r^3(x)), \\
\theta_{12}(x, y) &:= K_0(r^2(x), y, r^3(x)) \wedge E_1^*(y, r^3(x)), \\
\theta_{13}(x, y) &:= K_0(r^3(x), y, x) \wedge E_1^*(y, r^3(x)), \\
\theta_{14}(x, y) &:= K_0(r^3(x), y, x) \wedge E_2^*(y, r^3(x)) \wedge \neg E_1^*(y, r^3(x)), \\
\theta_{15}(x, y) &:= K_0(r^3(x), y, x) \wedge E_2(x, y) \wedge \neg E_1(x, y), \\
\theta_{16}(x, y) &:= K_0(r^3(x), y, x) \wedge E_1(x, y)
\end{aligned}$$

and the following holds for any $a \in M$:

$$K_0(\theta_0(a, M), \theta_1(a, M), \theta_2(a, M), \dots, \theta_{14}(a, M), \theta_{15}(a, M), \theta_{16}(a, M)).$$

It can be proved that the algebra $\mathfrak{P}_{M'_{1,4,4}}$ is non-commutative and strictly 5-deterministic.

Theorem 2. *The algebra $\mathfrak{P}_{M'_{s,4,4}}$ of binary isolating formulas with piecewise monotonic-to-left function r has $8s+9$ labels, is non-commutative and strictly $(2s+3)$ -deterministic for every $s \geq 1$.*

Proof. We assert that the algebra $\mathfrak{P}_{M'_{s,4,4}}$ has $8s+9$ binary isolating formulas:

$$\begin{aligned}
\theta_0(x, y) &:= x = y, \\
\theta_1(x, y) &:= K_0(x, y, r(x)) \wedge E_1(x, y), \\
\theta_{l_1}(x, y) &:= K_0(x, y, r(x)) \wedge E_{l_1}(x, y) \wedge \neg E_{l_1-1}(x, y), \text{ where } 2 \leq l_1 \leq s+1, \\
\theta_{l_2}(x, y) &:= K_0(x, y, r(x)) \wedge E_{2s+3-l_2}^*(y, r(x)) \wedge \neg E_{2s+2-l_2}^*(y, r(x)), \\
&\text{where } s+2 \leq l_2 \leq 2s+1, \\
\theta_{2s+2}(x, y) &:= K_0(x, y, r(x)) \wedge E_1^*(y, r(x)), \\
\theta_{2s+3}(x, y) &:= K_0(r(x), y, r^2(x)) \wedge E_1^*(y, r(x)), \\
\theta_{l_3}(x, y) &:= K_0(r(x), y, r^2(x)) \wedge E_{l_3-(2s+2)}^*(y, r(x)) \wedge \neg E_{l_3-(2s+3)}^*(y, r(x)), \\
&\text{where } 2s+4 \leq l_3 \leq 3s+3, \\
\theta_{l_4}(x, y) &:= K_0(r(x), y, r^2(x)) \wedge E_{4s+5-l_4}^*(y, r^2(x)) \wedge \neg E_{4s+4-l_4}^*(y, r^2(x)), \\
&\text{where } 3s+4 \leq l_4 \leq 4s+3, \\
\theta_{4s+4}(x, y) &:= K_0(r(x), y, r^2(x)) \wedge E_1^*(y, r^2(x)), \\
\theta_{4s+5}(x, y) &:= K_0(r^2(x), y, r^3(x)) \wedge E_1^*(y, r^2(x)), \\
\theta_{l_5}(x, y) &:= K_0(r^2(x), y, r^3(x)) \wedge E_{l_5-(4s+4)}^*(y, r^2(x)) \wedge \neg E_{l_5-(4s+5)}^*(y, r^2(x)), \\
&\text{where } 4s+6 \leq l_5 \leq 5s+5, \\
\theta_{l_6}(x, y) &:= K_0(r^2(x), y, r^3(x)) \wedge E_{6s+7-l_6}^*(y, r^3(x)) \wedge \neg E_{6s+6-l_6}^*(y, r^3(x)), \\
&\text{where } 5s+6 \leq l_6 \leq 6s+5, \\
\theta_{6s+6}(x, y) &:= K_0(r^2(x), y, r^3(x)) \wedge E_1^*(y, r^3(x)), \\
\theta_{6s+7}(x, y) &:= K_0(r^3(x), y, x) \wedge E_1^*(y, r^3(x)), \\
\theta_{l_7}(x, y) &:= K_0(r^3(x), y, x) \wedge E_{l_7-(6s+6)}^*(y, r^3(x)) \wedge \neg E_{l_7-(6s+7)}^*(y, r^3(x)), \\
&\text{where } 6s+8 \leq l_7 \leq 7s+7,
\end{aligned}$$

$$\theta_{l_8}(x, y) := K_0(r^3(x), y, x) \wedge E_{8s+9-l_8}(y, x) \wedge \neg E_{8s+8-l_8}(y, x),$$

$$\text{where } 7s + 8 \leq l_8 \leq 8s + 7,$$

$$\theta_{8s+8}(x, y) := K_0(r^3(x), y, x) \wedge E_1(x, y).$$

Thus, we have $2 + 2s + 2 + 2s + 2 + 2s + 2 + 2s + 1 = 8s + 9$ binary isolating formulas. Moreover, we have defined the formulas so that for any $a \in M$ the following holds:

$$K_0(\theta_0(a, M), \theta_1(a, M), \theta_2(a, M), \dots, \theta_{8s+6}(a, M), \theta_{8s+7}(a, M), \theta_{8s+8}(a, M)).$$

Prove now that the algebra $\mathfrak{P}_{M', 4, 4}$ is non-commutative and strictly $(2s + 3)$ -deterministic for every $s \geq 1$.

Firstly, obviously that $0 \cdot l = l \cdot 0 = \{l\}$ for any $0 \leq l \leq 8s + 8$.

Suppose further that $l_1 > 0$ and $l_2 > 0$. Consider the following formula:

$$\exists t[\theta_{l_1}(x, t) \wedge \theta_{l_2}(t, y)].$$

Case 1: $l_1 = 1$.

We have: $K_0(x, t, r(x))$ and $E_1(x, t)$. Let also $l_2 = 1$, i.e. $K_0(t, y, r(t))$ and $E_1(t, y)$, whence we obtain: $K_0(x, y, r(x)) \wedge E_1(x, y)$, i.e. $l_1 \cdot l_2 = \{l_2\}$.

Let now $2 \leq l_2 \leq s + 1$, i.e. $K_0(t, y, r(t)) \wedge E_{l_2}(t, y) \wedge \neg E_{l_2-1}(t, y)$. We have: $l_1 \cdot l_2 = \{l_2\}$. Similarly, we have that $l_2 \cdot l_1 = \{l_2\}$.

Let now $s + 2 \leq l_2 \leq 2s + 1$, i.e. $K_0(t, y, r(t)) \wedge E_{2s+3-l_2}^*(y, r(t)) \wedge \neg E_{2s+2-l_2}^*(y, r(t))$. We have: $l_1 \cdot l_2 = \{l_2\}$. Similarly, we have that $l_2 \cdot l_1 = \{l_2\}$.

Let now $l_2 = 2s + 2$, i.e. $K_0(t, y, r(t)) \wedge E_1^*(y, r(t))$. We have: $l_1 \cdot l_2 = \{2s + 2\}$. Consider $l_2 \cdot l_1$. We have that $l_2 \cdot l_1 = \{2s + 2, 2s + 3\}$.

Thus, we conclude that $\mathfrak{P}_{M', 4, 4}$ is not commutative for every $s \geq 1$.

Let now $l_2 = 2s + 3$, i.e. $K_0(r(t), y, r^2(t)) \wedge E_1^*(y, r(t))$. We have: $l_1 \cdot l_2 = \{2s + 2, 2s + 3\}$. Consider $l_2 \cdot l_1$. We have that $l_2 \cdot l_1 = \{2s + 3\}$.

Let now $2s + 4 \leq l_2 \leq 3s + 3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E_{l_2-(2s+2)}^*(y, r(t)) \wedge \neg E_{l_2-(2s+3)}^*(y, r(t)).$$

We have: $l_1 \cdot l_2 = \{l_2\}$. Similarly, we have that $l_2 \cdot l_1 = \{l_2\}$.

Let now $3s + 4 \leq l_2 \leq 4s + 3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E_{4s+5-l_2}^*(y, r^2(t)) \wedge \neg E_{4s+4-l_2}^*(y, r^2(t)).$$

We have: $l_1 \cdot l_2 = \{l_2\}$. Similarly, we have that $l_2 \cdot l_1 = \{l_2\}$.

Let now $l_2 = 4s + 4$, i.e. $K_0(r(t), y, r^2(t)) \wedge E_1^*(y, r^2(t))$. We have: $l_1 \cdot l_2 = \{4s + 4, 4s + 5\}$. Consider $l_2 \cdot l_1$. We also have that $l_2 \cdot l_1 = \{4s + 4, 4s + 5\}$.

Let now $l_2 = 4s + 5$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^2(t))$. We have: $l_1 \cdot l_2 = \{4s + 5\}$. Consider $l_2 \cdot l_1$. We also have that $l_2 \cdot l_1 = \{4s + 5\}$.

Let now $4s + 6 \leq l_2 \leq 5s + 5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E_{l_2-(4s+4)}^*(y, r^2(t)) \wedge \neg E_{l_2-(4s+5)}^*(y, r^2(t)).$$

We have: $l_1 \cdot l_2 = \{l_2\}$. Similarly, we have that $l_2 \cdot l_1 = \{l_2\}$.

Let now $5s + 6 \leq l_2 \leq 6s + 5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E_{6s+7-l_2}^*(y, r^3(t)) \wedge \neg E_{6s+6-l_2}^*(y, r^3(t)).$$

We have: $l_1 \cdot l_2 = \{l_2\}$. Similarly, we have that $l_2 \cdot l_1 = \{l_2\}$.

Let now $l_2 = 6s + 6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^3(t))$. We have: $l_1 \cdot l_2 = \{6s + 6\}$. Consider $l_2 \cdot l_1$. We have that $l_2 \cdot l_1 = \{6s + 6, 6s + 7\}$.

Let now $l_2 = 6s + 7$, i.e. $K_0(r^3(t), y, t) \wedge E_1^*(y, r^3(t))$. We have: $l_1 \cdot l_2 = \{6s + 6, 6s + 7\}$. Consider $l_2 \cdot l_1$. We have that $l_2 \cdot l_1 = \{6s + 7\}$.

Let now $6s + 8 \leq l_2 \leq 7s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{l_2-(6s+6)}^*(y, r^3(t)) \wedge \neg E_{l_2-(6s+7)}^*(y, r^3(t)).$$

We have: $l_1 \cdot l_2 = \{l_2\}$. Similarly, we have that $l_2 \cdot l_1 = \{l_2\}$.

Let now $7s + 8 \leq l_2 \leq 8s + 7$, i.e. $K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}(y, t) \wedge \neg E_{8s+8-l_2}(y, t)$. We have: $l_1 \cdot l_2 = \{l_2\}$. Similarly, we have that $l_2 \cdot l_1 = \{l_2\}$.

Let now $l_2 = 8s + 8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(t, y)$. We have: $l_1 \cdot l_2 = \{0, 1, 8s + 8\}$. Consider $l_2 \cdot l_1$. We have that $l_2 \cdot l_1 = \{0, 1, 8s + 8\}$.

Case 2: $2 \leq l_1 \leq s + 1$.

We have: $K_0(x, t, r(x)) \wedge E_{l_1}(x, t) \wedge \neg E_{l_1-1}(x, t)$.

Let also $2 \leq l_2 \leq s + 1$, i.e. $K_0(t, y, r(t)) \wedge E_{l_2}(t, y) \wedge \neg E_{l_2-1}(t, y)$. Then we have the following: if $l_1 \geq l_2$ then $l_1 \cdot l_2 = \{l_1\}$; if $l_1 < l_2$ then $l_1 \cdot l_2 = \{l_2\}$.

Let now $s + 2 \leq l_2 \leq 2s + 1$, i.e.

$$K_0(t, y, r(t)) \wedge E_{2s+3-l_2}^*(y, r(t)) \wedge \neg E_{2s+2-l_2}(y, r(t)).$$

We have the following: if $l_1 < 2s + 3 - l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2\}$.

If $l_1 = 2s + 3 - l_2$ then $l_1 \cdot l_2 = \{l_2\}$ and $l_2 \cdot l_1 = \{l_2, \dots, 4s + 5 - l_2\}$.

If $l_1 > 2s + 3 - l_2$ then $l_1 \cdot l_2 = \{2s + 3 - l_1\}$ and $l_2 \cdot l_1 = \{2s + 2 + l_1\}$.

Let now $l_2 = 2s + 2$, i.e. $K_0(t, y, r(t)) \wedge E_1^*(y, r(t))$. Then $l_1 \cdot l_2 = \{2s + 3 - l_1\}$ and $l_2 \cdot l_1 = \{2s + 2 + l_1\}$. Let now $l_2 = 2s + 3$, i.e. $K_0(r(t), y, r^2(t)) \wedge E_1^*(y, r(t))$. Also, then $l_1 \cdot l_2 = \{2s + 3 - l_1\}$ and $l_2 \cdot l_1 = \{2s + 2 + l_1\}$.

Let now $2s + 4 \leq l_2 \leq 3s + 3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E_{l_2-(2s+2)}^*(y, r(t)) \wedge \neg E_{l_2-(2s+3)}^*(y, r(t)).$$

We have the following: if $l_1 < l_2 - (2s + 2)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2\}$.

If $l_1 = l_2 - (2s + 2)$ then $l_1 \cdot l_2 = \{2s + 3 - l_1, \dots, l_2\}$ and $l_2 \cdot l_1 = \{l_2\}$.

If $l_1 > l_2 - (2s + 2)$ then $l_1 \cdot l_2 = \{2s + 3 - l_1\}$ and $l_2 \cdot l_1 = \{2s + 2 + l_1\}$.

Let now $3s + 4 \leq l_2 \leq 4s + 3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E_{4s+5-l_2}^*(y, r^2(t)) \wedge \neg E_{4s+4-l_2}^*(y, r^2(t)).$$

We have the following: if $l_1 < 4s + 5 - l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2\}$.

If $l_1 = 4s + 5 - l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2, \dots, 8s + 9 - l_2\}$.

If $l_1 > 4s + 5 - l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s + 4 + l_1\}$.

Let now $l_2 = 4s + 4$, i.e. $K_0(r(t), y, r^2(t)) \wedge E_1^*(y, r^2(t))$. Then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s + 4 + l_1\}$. Let now $l_2 = 4s + 5$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^2(t))$. Also, then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s + 4 + l_1\}$.

Let now $4s + 6 \leq l_2 \leq 5s + 5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E_{l_2-(4s+4)}^*(y, r^2(t)) \wedge \neg E_{l_2-(4s+5)}^*(y, r^2(t)).$$

We have the following: if $l_1 \leq l_2 - (4s + 4)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2\}$.

If $l_1 > l_2 - (4s + 4)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s + 4 + l_1\}$.

Let now $5s + 6 \leq l_2 \leq 6s + 5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E_{6s+7-l_2}^*(y, r^3(t)) \wedge \neg E_{6s+6-l_2}^*(y, r^3(t)).$$

We have the following: if $l_1 < 6s + 7 - l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2\}$.

If $l_1 = 6s + 7 - l_2$ then $l_1 \cdot l_2 = \{l_2\}$ and $l_2 \cdot l_1 = \{l_2, \dots, 6s + 6 + l_1\}$.

If $l_1 > 6s + 7 - l_2$ then $l_1 \cdot l_2 = \{6s + 7 - l_1\}$ and $l_2 \cdot l_1 = \{6s + 6 + l_1\}$.

Let now $l_2 = 6s + 6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^3(t))$. Then $l_1 \cdot l_2 = \{6s + 7 - l_1\}$ and $l_2 \cdot l_1 = \{6s + 6 + l_1\}$.

Let now $l_2 = 6s + 7$, i.e. $K_0(r^3(t), y, t) \wedge E_1^*(y, r^3(t))$. Then also $l_1 \cdot l_2 = \{6s + 7 - l_1\}$ and $l_2 \cdot l_1 = \{6s + 6 + l_1\}$.

Let now $6s + 8 \leq l_2 \leq 7s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{l_2-(6s+6)}^*(y, r^3(t)) \wedge \neg E_{l_2-(6s+7)}^*(y, r^3(t)).$$

We have the following: if $l_1 < l_2 - (6s + 6)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2\}$.

If $l_1 = l_2 - (6s + 6)$ then $l_1 \cdot l_2 = \{6s + 7 - l_1, \dots, l_2\}$ and $l_2 \cdot l_1 = \{l_2\}$.

If $l_1 > l_2 - (6s + 6)$ then $l_1 \cdot l_2 = \{6s + 7 - l_1\}$ and $l_2 \cdot l_1 = \{6s + 6 + l_1\}$.

Let now $7s + 8 \leq l_2 \leq 8s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}(y, t) \wedge \neg E_{8s+8-l_2}(y, t).$$

We have the following: if $l_1 < 8s + 9 - l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2\}$.

If $l_1 = 8s + 9 - l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2, \dots, 8s + 8, 0, 1, \dots, l_1\}$. If

$l_1 > 8s + 9 - l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$. Let now $l_2 = 8s + 8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(t, y)$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$.

Case 3. $s + 2 \leq l_1 \leq 2s + 1$.

We have: $K_0(x, t, r(x)) \wedge E_{2s+3-l_1}^*(t, r(x)) \wedge \neg E_{2s+2-l_1}^*(t, r(x))$.

Let also $s + 2 \leq l_2 \leq 2s + 1$, i.e.

$$K_0(t, y, r(t)) \wedge E_{2s+3-l_2}^*(y, r(t)) \wedge \neg E_{2s+2-l_2}^*(y, r(t)).$$

Then we have the following: if $l_1 < l_2$ then $l_1 \cdot l_2 = \{6s + 7 - l_1\}$ and $l_2 \cdot l_1 = \{2s + 2 + l_1\}$. If $l_1 = l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{2s + 2 + l_1, \dots, 6s + 7 - l_1\}$.

If $l_1 > l_2$ then $l_1 \cdot l_2 = \{2s + 2 + l_2\}$ and $l_2 \cdot l_1 = \{6s + 7 - l_2\}$.

Let now $l_2 = 2s + 2$, i.e. $K_0(t, y, r(t)) \wedge E_1^*(y, r(t))$. Then we have $l_1 \cdot l_2 = \{6s + 7 - l_1\}$ and $l_2 \cdot l_1 = \{2s + 2 + l_1\}$.

Let now $l_2 = 2s + 3$, i.e. $K_0(r(t), y, r^2(t)) \wedge E_1^*(y, r(t))$. Then we also have $l_1 \cdot l_2 = \{6s + 7 - l_1\}$ and $l_2 \cdot l_1 = \{2s + 2 + l_1\}$.

Let now $2s + 4 \leq l_2 \leq 3s + 3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E_{l_2-(2s+2)}^*(y, r(t)) \wedge \neg E_{l_2-(2s+3)}^*(y, r(t)).$$

We have the following: if $2s + 3 - l_1 = l_2 - (2s + 2)$ then $l_1 \cdot l_2 = \{6s + 7 - l_1\}$ and $l_2 \cdot l_1 = \{2s + 2 + l_1, \dots, 6s + 7 - l_1\}$. If $2s + 3 - l_1 < l_2 - (2s + 2)$ then $l_1 \cdot l_2 = \{2s + 2 + l_2\}$ and $l_2 \cdot l_1 = \{6s + 7 - l_2\}$. If $2s + 3 - l_1 > l_2 - (2s + 2)$ then $l_1 \cdot l_2 = \{6s + 7 - l_1\}$ and $l_2 \cdot l_1 = \{2s + 2 + l_1\}$.

Let now $3s + 4 \leq l_2 \leq 4s + 3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E_{4s+5-l_2}^*(y, r^2(t)) \wedge \neg E_{4s+4-l_2}^*(y, r^2(t)).$$

We have the following: if $2s+3-l_1 = 4s+5-l_2$ then $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{2s+2+l_1, \dots, 10s+11-l_2\}$.

If $2s+3-l_1 < 4s+5-l_2$ then $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$.

If $2s+3-l_1 > 4s+5-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4+l_1\}$.

Let now $l_2 = 4s+4$, i.e. $K_0(r(t), y, r^2(t)) \wedge E_1^*(y, r^2(t))$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4+l_1\}$.

Let now $l_2 = 4s+5$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^2(t))$. Then we also have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4+l_1\}$.

Let now $4s+6 \leq l_2 \leq 5s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E_{l_2-(4s+4)}^*(y, r^2(t)) \wedge \neg E_{l_2-(4s+5)}^*(y, r^2(t)).$$

We have the following: if $2s+3-l_1 > l_2-(4s+4)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4+l_1\}$. If $2s+3-l_1 = l_2-(4s+4)$ then $l_1 \cdot l_2 = \{4s+4+l_1, \dots, 2s+2+l_2\}$ and $l_2 \cdot l_1 = \{4s+4+l_1\}$. If $2s+3-l_1 < l_2-(4s+4)$ then $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$.

Let now $5s+6 \leq l_2 \leq 6s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E_{6s+7-l_2}^*(y, r^3(t)) \wedge \neg E_{6s+6-l_2}^*(y, r^3(t)).$$

We have the following: if $2s+3-l_1 > 6s+7-l_2$ then $l_1 \cdot l_2 = \{2s+3-l_1\}$ and $l_2 \cdot l_1 = \{6s+6+l_1\}$. If $2s+3-l_1 = 6s+7-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{2s+2+l_2, \dots, 8s+8, 0, 1, \dots, 6s+7-l_2\}$. If $2s+3-l_1 < 6s+7-l_2$ then $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{6s+7-l_2\}$.

Let now $l_2 = 6s+6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{2s+3-l_1\}$ and $l_2 \cdot l_1 = \{6s+6+l_1\}$.

Let now $l_2 = 6s+7$, i.e. $K_0(r^3(t), y, t) \wedge E_1^*(y, r^3(t))$. Then we also have $l_1 \cdot l_2 = \{2s+3-l_1\}$ and $l_2 \cdot l_1 = \{6s+6+l_1\}$.

Let now $6s+8 \leq l_2 \leq 7s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{l_2-(6s+6)}^*(y, r^3(t)) \wedge \neg E_{l_2-(6s+7)}^*(y, r^3(t)).$$

We have the following: if $2s+3-l_1 \geq l_2-(6s+6)$ then $l_1 \cdot l_2 = \{2s+3-l_1\}$ and $l_2 \cdot l_1 = \{6s+6+l_1\}$. If $2s+3-l_1 < l_2-(6s+6)$ then $l_1 \cdot l_2 = \{l_2-(6s+6)\}$ and $l_2 \cdot l_1 = \{14s+15-l_2\}$.

Let now $7s+8 \leq l_2 \leq 8s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}^*(y, t) \wedge \neg E_{8s+8-l_2}^*(y, t).$$

We have the following: if $2s+3-l_1 > 8s+9-l_2$ then $l_1 \cdot l_2 = \{2s+3-l_1\}$ and $l_2 \cdot l_1 = \{l_1\}$. If $2s+3-l_1 = 8s+9-l_2$ then $l_1 \cdot l_2 = \{l_1\}$ and $l_2 \cdot l_1 = \{l_1, \dots, 4s+5-l_1\}$. If $2s+3-l_1 < 8s+9-l_2$ then $l_1 \cdot l_2 = \{l_2-(6s+6)\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$.

Let now $l_2 = 8s+8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$.

Case 4. $l_1 = 2s+2$.

We have: $K_0(x, t, r(x)) \wedge E_1^*(t, r(x))$.

Let also $l_2 = 2s + 2$, i.e. $K_0(t, y, r(t)) \wedge E_1^*(y, r(t))$. Then we have $l_1 \cdot l_2 = \{4s + 4, 4s + 5\}$. Let now $l_2 = 2s + 3$, i.e. $K_0(r(t), y, r^2(t)) \wedge E_1^*(y, r(t))$. Then we have $l_1 \cdot l_2 = \{4s + 5\}$ and $l_2 \cdot l_1 = \{4s + 4\}$.

Let now $2s + 4 \leq l_2 \leq 3s + 3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E_{l_2-(2s+2)}^*(y, r(t)) \wedge \neg E_{l_2-(2s+3)}^*(y, r(t)).$$

We have the following: $l_1 \cdot l_2 = \{2s + 2 + l_2\}$ and $l_2 \cdot l_1 = \{6s + 7 - l_2\}$.

Let now $3s + 4 \leq l_2 \leq 4s + 3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E_{4s+5-l_2}^*(y, r^2(t)) \wedge \neg E_{4s+4-l_2}^*(y, r^2(t)).$$

We have the following: $l_1 \cdot l_2 = \{2s + 2 + l_2\}$ and $l_2 \cdot l_1 = \{10s + 11 - l_2\}$.

Let now $l_2 = 4s + 4$, i.e. $K_0(r(t), y, r^2(t)) \wedge E_1^*(y, r^2(t))$. Then we have $l_1 \cdot l_2 = \{6s + 6\}$ and $l_2 \cdot l_1 = \{6s + 6, 6s + 7\}$.

Let now $l_2 = 4s + 5$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^2(t))$. Then we have $l_1 \cdot l_2 = \{6s + 6, 6s + 7\}$ and $l_2 \cdot l_1 = \{6s + 6\}$.

Let now $4s + 6 \leq l_2 \leq 5s + 5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E_{l_2-(4s+4)}^*(y, r^2(t)) \wedge \neg E_{l_2-(4s+5)}^*(y, r^2(t)).$$

We have the following: $l_1 \cdot l_2 = \{2s + 2 + l_2\}$ and $l_2 \cdot l_1 = \{10s + 11 - l_2\}$.

Let now $5s + 6 \leq l_2 \leq 6s + 5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E_{6s+7-l_2}^*(y, r^3(t)) \wedge \neg E_{6s+6-l_2}^*(y, r^3(t)).$$

We have the following: $l_1 \cdot l_2 = \{2s + 2 + l_2\}$ and $l_2 \cdot l_1 = \{6s + 7 - l_2\}$.

Let now $l_2 = 6s + 6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^3(t))$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{8s + 8, 0, 1\}$. Let now $l_2 = 6s + 7$, i.e. $K_0(r^3(t), y, t) \wedge E_1^*(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{1\}$ and $l_2 \cdot l_1 = \{8s + 8\}$.

Let now $6s + 8 \leq l_2 \leq 7s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{l_2-(6s+6)}^*(y, r^3(t)) \wedge \neg E_{l_2-(6s+7)}^*(y, r^3(t)).$$

We have the following: $l_1 \cdot l_2 = \{l_2 - (6s + 6)\}$ and $l_2 \cdot l_1 = \{14s + 15 - l_2\}$.

Let now $7s + 8 \leq l_2 \leq 8s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}(y, t) \wedge \neg E_{8s+8-l_2}(y, t).$$

We have the following: $l_1 \cdot l_2 = \{l_2 - (6s + 6)\}$ and $l_2 \cdot l_1 = \{10s + 11 - l_2\}$.

Let now $l_2 = 8s + 8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = \{2s + 2\}$ and $l_2 \cdot l_1 = \{2s + 2, 2s + 3\}$.

Case 5. $l_1 = 2s + 3$.

We have: $K_0(r(x), t, r^2(x)) \wedge E_1^*(t, r(x))$.

Let also $l_2 = 2s + 3$, i.e. $K_0(r(t), y, r^2(t)) \wedge E_1^*(y, r(t))$. Then we have $l_1 \cdot l_2 = \{4s + 4, 4s + 5\}$. Let now $2s + 4 \leq l_2 \leq 3s + 3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E_{l_2-(2s+2)}^*(y, r(t)) \wedge \neg E_{l_2-(2s+3)}^*(y, r(t)).$$

We have the following: $l_1 \cdot l_2 = \{2s + 2 + l_2\}$ and $l_2 \cdot l_1 = \{6s + 7 - l_2\}$.

Let now $3s + 4 \leq l_2 \leq 4s + 3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E_{4s+5-l_2}^*(y, r^2(t)) \wedge \neg E_{4s+4-l_2}^*(y, r^2(t)).$$

We have the following: $l_1 \cdot l_2 = \{2s + 2 + l_2\}$ and $l_2 \cdot l_1 = \{10s + 11 - l_2\}$.

Let now $l_2 = 4s + 4$, i.e. $K_0(r(t), y, r^2(t)) \wedge E_1^*(y, r^2(t))$. Then we have $l_1 \cdot l_2 = \{6s + 6, 6s + 7\}$ and $l_2 \cdot l_1 = \{6s + 7\}$.

Let now $l_2 = 4s + 5$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^2(t))$. Then we have $l_1 \cdot l_2 = \{6s + 7\}$ and $l_2 \cdot l_1 = \{6s + 6, 6s + 7\}$.

Let now $4s + 6 \leq l_2 \leq 5s + 5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E_{l_2-(4s+4)}^*(y, r^2(t)) \wedge \neg E_{l_2-(4s+5)}^*(y, r^2(t)).$$

We have the following: $l_1 \cdot l_2 = \{2s + 2 + l_2\}$ and $l_2 \cdot l_1 = \{10s + 11 - l_2\}$.

Let now $5s + 6 \leq l_2 \leq 6s + 5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E_{6s+7-l_2}^*(y, r^3(t)) \wedge \neg E_{6s+6-l_2}^*(y, r^3(t)).$$

We have the following: $l_1 \cdot l_2 = \{2s + 2 + l_2\}$ and $l_2 \cdot l_1 = \{6s + 7 - l_2\}$.

Let now $l_2 = 6s + 6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{8s + 8\}$ and $l_2 \cdot l_1 = \{1\}$.

Let now $l_2 = 6s + 7$, i.e. $K_0(r^3(t), y, t) \wedge E_1^*(y, r^3(t))$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{8s + 8, 0, 1\}$. Let now $6s + 8 \leq l_2 \leq 7s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{l_2-(6s+6)}^*(y, r^3(t)) \wedge \neg E_{l_2-(6s+7)}^*(y, r^3(t)).$$

We have the following: $l_1 \cdot l_2 = \{l_2 - (6s + 6)\}$ and $l_2 \cdot l_1 = \{14s + 15 - l_2\}$.

Let now $7s + 8 \leq l_2 \leq 8s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}(y, t) \wedge \neg E_{8s+8-l_2}(y, t).$$

We have the following: $l_1 \cdot l_2 = \{l_2 - (6s + 6)\}$ and $l_2 \cdot l_1 = \{10s + 11 - l_2\}$.

Let now $l_2 = 8s + 8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = \{2s + 2, 2s + 3\}$ and $l_2 \cdot l_1 = \{2s + 3\}$.

Case 6. $2s + 4 \leq l_1 \leq 3s + 3$.

We have: $K_0(r(x), t, r^2(x)) \wedge E_{l_1-(2s+2)}^*(t, r(x)) \wedge \neg E_{l_1-(2s+3)}^*(t, r(x))$.

Let also $2s + 4 \leq l_2 \leq 3s + 3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E_{l_2-(2s+2)}^*(y, r(t)) \wedge \neg E_{l_2-(2s+3)}^*(y, r(t)).$$

We have the following: if $l_1 = l_2$ then $l_1 \cdot l_2 = \{6s + 7 - l_2, \dots, 2s + 2 + l_2\}$.

If $l_1 < l_2$ then $l_1 \cdot l_2 = \{2s + 2 + l_2\}$ and $l_2 \cdot l_1 = \{6s + 7 - l_2\}$.

If $l_1 > l_2$ then $l_1 \cdot l_2 = \{6s + 7 - l_1\}$ and $l_2 \cdot l_1 = \{2s + 2 + l_1\}$.

Let now $3s + 4 \leq l_2 \leq 4s + 3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E_{4s+5-l_2}^*(y, r^2(t)) \wedge \neg E_{4s+4-l_2}^*(y, r^2(t)).$$

We have the following: if $l_1 - (2s + 2) < 4s + 5 - l_2$ then $l_1 \cdot l_2 = \{2s + 2 + l_2\}$ and $l_2 \cdot l_1 = \{10s + 11 - l_2\}$. If $l_1 - (2s + 2) = 4s + 5 - l_2$ then $l_1 \cdot l_2 = \{2s + 2 + l_2, \dots, 10s + 11 - l_2\}$ and $l_2 \cdot l_1 = \{4s + 4 + l_2\}$. If $l_1 - (2s + 2) > 4s + 5 - l_2$ then $l_1 \cdot l_2 = \{4s + 4 + l_1\}$ and $l_2 \cdot l_1 = \{8s + 9 - l_1\}$.

Let now $l_2 = 4s + 4$, i.e. $K_0(r(t), y, r^2(t)) \wedge E_1^*(y, r^2(t))$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s + 4 + l_1\}$.

Let now $l_2 = 4s + 5$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^2(t))$. Then we also have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s + 4 + l_1\}$.

Let now $4s + 6 \leq l_2 \leq 5s + 5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E_{l_2-(4s+4)}^*(y, r^2(t)) \wedge \neg E_{l_2-(4s+5)}^*(y, r^2(t)).$$

We have the following: if $l_1 - (2s+2) < l_2 - (4s+4)$ then $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$. If $l_1 - (2s+2) = l_2 - (4s+4)$ then $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{10s+11-l_2, \dots, 2s+2+l_2\}$. If $l_1 - (2s+2) > l_2 - (4s+4)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4+l_1\}$.

Let now $5s + 6 \leq l_2 \leq 6s + 5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E_{6s+7-l_2}^*(y, r^3(t)) \wedge \neg E_{6s+6-l_2}^*(y, r^3(t)).$$

We have the following: if $l_1 - (2s+2) \leq 6s+7-l_2$ then $l_1 \cdot l_2 = \{2s+2+l_2\}$ and $l_2 \cdot l_1 = \{6s+7-l_2\}$. If $l_1 - (2s+2) > 6s+7-l_2$ then $l_1 \cdot l_2 = \{10s+11-l_1\}$ and $l_2 \cdot l_1 = \{l_1 - (2s+2)\}$.

Let now $l_2 = 6s + 6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{10s+11-l_1\}$ and $l_2 \cdot l_1 = \{l_1 - (2s+2)\}$. Let now $l_2 = 6s + 7$, i.e. $K_0(r^3(t), y, t) \wedge E_1^*(y, r^3(t))$. Then we also have $l_1 \cdot l_2 = \{10s+11-l_1\}$ and $l_2 \cdot l_1 = \{l_1 - (2s+2)\}$. Let now $6s + 8 \leq l_2 \leq 7s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{l_2-(6s+6)}^*(y, r^3(t)) \wedge \neg E_{l_2-(6s+7)}^*(y, r^3(t)).$$

We have the following: if $l_1 - (2s+2) < l_2 - (6s+6)$ then $l_1 \cdot l_2 = \{l_2 - (6s+6)\}$ and $l_2 \cdot l_1 = \{14s+15-l_2\}$. If $l_1 - (2s+2) = l_2 - (6s+6)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{10s+11-l_1, \dots, 0, \dots, l_1 - (2s+2)\}$. If $l_1 - (2s+2) > l_2 - (6s+6)$ then $l_1 \cdot l_2 = \{10s+11-l_1\}$ and $l_2 \cdot l_1 = \{l_1 - (2s+2)\}$.

Let now $7s + 8 \leq l_2 \leq 8s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}(y, t) \wedge \neg E_{8s+8-l_2}(y, t).$$

We have the following: if $l_1 - (2s+2) < 8s+9-l_2$ then $l_1 \cdot l_2 = \{l_2 - (6s+6)\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$. If $l_1 - (2s+2) = 8s+9-l_2$ then $l_1 \cdot l_2 = \{4s+5-l_1, \dots, l_1\}$ and $l_2 \cdot l_1 = \{l_1\}$. If $l_1 - (2s+2) > 8s+9-l_2$ then $l_1 \cdot l_2 = \{10s+11-l_1\}$ and $l_2 \cdot l_1 = \{l_1\}$.

Let now $l_2 = 8s + 8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$.

Case 7. $3s + 4 \leq l_1 \leq 4s + 3$.

We have: $K_0(r(x), t, r^2(x)) \wedge E_{4s+5-l_1}^*(t, r^2(x)) \wedge \neg E_{4s+4-l_1}^*(t, r^2(x))$.

Let also $3s + 4 \leq l_2 \leq 4s + 3$, i.e.

$$K_0(r(t), y, r^2(t)) \wedge E_{4s+5-l_2}^*(y, r^2(t)) \wedge \neg E_{4s+4-l_2}^*(y, r^2(t)).$$

We have the following: if $l_1 \leq l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4+l_1\}$.

If $l_1 > l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4+l_2\}$.

Let now $l_2 = 4s + 4$, i.e. $K_0(r(t), y, r^2(t)) \wedge E_1^*(y, r^2(t))$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4+l_1\}$.

Let now $l_2 = 4s + 5$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^2(t))$. Then we also have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4+l_1\}$. Let now $4s + 6 \leq l_2 \leq 5s + 5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E_{l_2-(4s+4)}^*(y, r^2(t)) \wedge \neg E_{l_2-(4s+5)}^*(y, r^2(t)).$$

We have the following: if $4s+5-l_1 < l_2-(4s+4)$ then $l_1 \cdot l_2 = \{14s+5-l_2\}$ and $l_2 \cdot l_1 = \{l_2-(4s+4)\}$. If $4s+5-l_1 = l_2-(4s+4)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4+l_1, \dots, 40, 0, \dots, 4s+5-l_1\}$. If $4s+5-l_1 > l_2-(4s+4)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s+4+l_1\}$.

Let now $5s+6 \leq l_2 \leq 6s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E_{6s+7-l_2}^*(y, r^3(t)) \wedge \neg E_{6s+6-l_2}^*(y, r^3(t)).$$

We have the following: if $4s+5-l_1 < 6s+7-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2-(4s+4)\}$. If $4s+5-l_1 = 6s+7-l_2$ then $l_1 \cdot l_2 = \{l_2-(4s+4), \dots, 6s+7-l_1\}$ and $l_2 \cdot l_1 = \{l_2-(4s+4)\}$. If $4s+5-l_1 > 6s+7-l_2$ then $l_1 \cdot l_2 = \{6s+7-l_1\}$ and $l_2 \cdot l_1 = \{l_1-(2s+2)\}$.

Let now $l_2 = 6s+6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{6s+7-l_1\}$ and $l_2 \cdot l_1 = \{l_1-(2s+2)\}$.

Let now $l_2 = 6s+7$, i.e. $K_0(r^3(t), y, t) \wedge E_1^*(y, r^3(t))$. Then we also have $l_1 \cdot l_2 = \{6s+7-l_1\}$ and $l_2 \cdot l_1 = \{l_1-(2s+2)\}$.

Let now $6s+8 \leq l_2 \leq 7s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{l_2-(6s+6)}^*(y, r^3(t)) \wedge \neg E_{l_2-(6s+7)}^*(y, r^3(t)).$$

We have the following: if $4s+5-l_1 < l_2-(6s+6)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2-(4s+4)\}$. If $4s+5-l_1 = l_2-(6s+6)$ then $l_1 \cdot l_2 = \{l_2-(4s+4)\}$ and $l_2 \cdot l_1 = \{8s+9-l_2, \dots, l_2-(4s+4)\}$. If $4s+5-l_1 > l_2-(6s+6)$ then $l_1 \cdot l_2 = \{6s+7-l_1\}$ and $l_2 \cdot l_1 = \{l_1-(2s+2)\}$.

Let now $7s+8 \leq l_2 \leq 8s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}(y, t) \wedge \neg E_{8s+8-l_2}(y, t).$$

We have the following: if $4s+5-l_1 < 8s+9-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2-(4s+4)\}$. If $4s+5-l_1 \geq 8s+9-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$.

Let now $l_2 = 8s+8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$.

Case 8. $l_1 = 4s+4$.

We have: $K_0(r(x), t, r^2(x)) \wedge E_1^*(t, r^2(x))$.

Let also $l_2 = 4s+4$, i.e. $K_0(r(t), y, r^2(t)) \wedge E_1^*(y, r^2(t))$. Then we have $l_1 \cdot l_2 = \{8s+8\}$. Let now $l_2 = 4s+5$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^2(t))$. Then we have $l_1 \cdot l_2 = \{0, 1, 8s+8\}$ and $l_2 \cdot l_1 = \{2s+2\}$.

Let now $4s+6 \leq l_2 \leq 5s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E_{l_2-(4s+4)}^*(y, r^2(t)) \wedge \neg E_{l_2-(4s+5)}^*(y, r^2(t)).$$

We have the following: $l_1 \cdot l_2 = \{6s+7-l_2\}$ and $l_2 \cdot l_1 = \{l_2-(4s+4)\}$.

Let now $5s+6 \leq l_2 \leq 6s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E_{6s+7-l_2}^*(y, r^3(t)) \wedge \neg E_{6s+6-l_2}^*(y, r^3(t)).$$

We have the following: $l_1 \cdot l_2 = \{l_2-(4s+4)\}$ and $l_2 \cdot l_1 = \{l_2-(3s+2)\}$.

Let now $l_2 = 6s+6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{2s+2, 2s+3\}$ and $l_2 \cdot l_1 = \{2s+2\}$.

Let now $l_2 = 6s+7$, i.e. $K_0(r^3(t), y, t) \wedge E_1^*(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{2s+3\}$ and $l_2 \cdot l_1 = \{2s+2, 2s+3\}$.

Let now $6s + 8 \leq l_2 \leq 7s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{l_2-(6s+6)}^*(y, r^3(t)) \wedge \neg E_{l_2-(6s+7)}^*(y, r^3(t)).$$

We have the following: $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2 - (4s + 4)\}$.

Let now $7s + 8 \leq l_2 \leq 8s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}(y, t) \wedge \neg E_{8s+8-l_2}(y, t).$$

We also have the following: $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2 - (4s + 4)\}$.

Let now $l_2 = 8s + 8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s + 4\}$.

Case 9. $l_1 = 4s + 5$.

We have: $K_0(r^2(x), t, r^3(x)) \wedge E_1^*(t, r^2(x))$.

Let also $l_2 = 4s + 5$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^2(t))$. Then we have $l_1 \cdot l_2 = \{1\}$. Let now $4s + 6 \leq l_2 \leq 5s + 5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E_{l_2-(4s+4)}^*(y, r^2(t)) \wedge \neg E_{l_2-(4s+5)}^*(y, r^2(t)).$$

We have the following: $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2 - (4s + 4)\}$.

Let now $5s + 6 \leq l_2 \leq 6s + 5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E_{6s+7-l_2}^*(y, r^3(t)) \wedge \neg E_{6s+6-l_2}^*(y, r^3(t)).$$

We have the following: $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2 - (4s + 4)\}$.

Let now $l_2 = 6s + 6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{2s + 2\}$ and $l_2 \cdot l_1 = \{2s + 2, 2s + 3\}$.

Let now $l_2 = 6s + 7$, i.e. $K_0(r^3(t), y, t) \wedge E_1^*(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{2s + 2, 2s + 3\}$ and $l_2 \cdot l_1 = \{2s + 3\}$.

Let now $6s + 8 \leq l_2 \leq 7s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{l_2-(6s+6)}^*(y, r^3(t)) \wedge \neg E_{l_2-(6s+7)}^*(y, r^3(t)).$$

We have the following: $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2 - (4s + 4)\}$.

Let now $7s + 8 \leq l_2 \leq 8s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}(y, t) \wedge \neg E_{8s+8-l_2}^*(y, t).$$

We have the following: $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2 - (4s + 4)\}$.

Let now $l_2 = 8s + 8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{4s + 4, 4s + 5\}$.

Case 10. $4s + 6 \leq l_1 \leq 5s + 5$.

We have: $K_0(r^2(x), t, r^3(x)) \wedge E_{l_1-(4s+4)}^*(t, r^2(x)) \wedge \neg E_{l_1-(4s+5)}^*(t, r^2(x))$.

Let also $4s + 6 \leq l_2 \leq 5s + 5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E_{l_2-(4s+4)}^*(y, r^2(t)) \wedge \neg E_{l_2-(4s+5)}^*(y, r^2(t)).$$

We have the following: if $l_1 < l_2$ then $l_1 \cdot l_2 = \{l_2 - (4s + 4)\}$, if $l_1 \geq l_2$ then $l_1 \cdot l_2 = \{l_1 - (4s + 4)\}$. Let now $5s + 6 \leq l_2 \leq 6s + 5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E_{6s+7-l_2}^*(y, r^3(t)) \wedge \neg E_{6s+6-l_2}^*(y, r^3(t)).$$

We have the following: if $l_1 - (4s+4) < 6s+7-l_2$ then $l_1 \cdot l_2 = \{l_2 - (4s+4)\}$,
if $l_1 - (4s+4) \geq 6s+7-l_2$ then $l_1 \cdot l_2 = \{l_1 - (3s+4)\}$.

Let now $l_2 = 6s+6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^3(t))$.

We have: $l_1 \cdot l_2 = \{l_1 - (3s+4)\}$ and $l_2 \cdot l_1 = \{l_1 - (2s+2)\}$.

Let now $l_2 = 6s+7$, i.e. $K_0(r^3(t), y, t) \wedge E_1^*(y, r^3(t))$.

We have: $l_1 \cdot l_2 = \{l_1 - (3s+4)\}$ and $l_2 \cdot l_1 = \{6s+7-l_1\}$.

Let now $6s+8 \leq l_2 \leq 7s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{l_2-(6s+6)}^*(y, r^3(t)) \wedge \neg E_{l_2-(6s+7)}^*(y, r^3(t)).$$

We have the following: if $l_1 - (4s+4) = l_2 - (6s+6)$ then $l_1 \cdot l_2 = \{l_1 - (4s+3), l_1 - (4s+2), \dots, l_1 - (2s+2)\}$ and $l_2 \cdot l_1 = \{l_2 - (4s+4)\}$.

If $l_1 - (4s+4) < l_2 - (6s+6)$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2 - (4s+4)\}$.

If $l_1 - (4s+4) > l_2 - (6s+6)$ then $l_1 \cdot l_2 = \{6s+7-l_1\}$ and $l_2 \cdot l_1 = \{l_2 - (2s+2)\}$.

Let now $7s+8 \leq l_2 \leq 8s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}^*(y, t) \wedge \neg E_{8s+8-l_2}^*(y, t).$$

We have the following: if $l_1 - (4s+4) = 8s+9-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{8s+9-l_1, 8s+8-l_1, \dots, l_1\}$.

If $l_1 - (4s+4) < 8s+9-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_2 - (4s+4)\}$.

If $l_1 - (4s+4) > 8s+9-l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$.

Let now $l_2 = 8s+8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$.

Case 11. $5s+6 \leq l_1 \leq 6s+5$.

We have: $K_0(r^2(x), t, r^3(x)) \wedge E_{6s+7-l_1}^*(t, r^2(x)) \wedge \neg E_{6s+6-l_1}^*(t, r^2(x))$.

Let also $5s+6 \leq l_2 \leq 6s+5$, i.e.

$$K_0(r^2(t), y, r^3(t)) \wedge E_{6s+7-l_2}^*(y, r^2(t)) \wedge \neg E_{6s+6-l_2}^*(y, r^2(t)).$$

We have the following: if $l_1 = l_2$ then $l_1 \cdot l_2 = \{10s+11-l_1\}$. If $l_1 < l_2$ then $l_1 \cdot l_2 = \{10s+11-l_1\}$. If $l_1 > l_2$ then $l_1 \cdot l_2 = \{l_1 - (2s+2)\}$.

Let now $l_2 = 6s+6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^3(t))$.

We have: $l_1 \cdot l_2 = \{10s+11-l_1\}$ and $l_2 \cdot l_1 = \{l_1 - (2s+2)\}$.

Let now $l_2 = 6s+7$, i.e. $K_0(r^3(t), y, t) \wedge E_1^*(y, r^3(t))$.

We have: $l_1 \cdot l_2 = \{10s+11-l_1\}$ and $l_2 \cdot l_1 = \{l_1 - (2s+2)\}$.

Let now $6s+8 \leq l_2 \leq 7s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{l_2-(6s+6)}^*(y, r^3(t)) \wedge \neg E_{l_2-(6s+7)}^*(y, r^3(t)).$$

We have the following: if $6s+7-l_1 = l_2 - (6s+6)$ then $l_1 \cdot l_2 = \{10s+11-l_1\}$ and $l_2 \cdot l_1 = \{l_1 - (2s+2)\}$. If $6s+7-l_1 < l_2 - (6s+6)$ then $l_1 \cdot l_2 = \{l_2 - (2s+2)\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$. If $6s+7-l_1 > l_2 - (6s+6)$ then $l_1 \cdot l_2 = \{10s+11-l_1\}$ and $l_2 \cdot l_1 = \{l_1 - (2s+2)\}$.

Let now $7s+8 \leq l_2 \leq 8s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}^*(y, t) \wedge \neg E_{8s+8-l_2}^*(y, t).$$

We have the following: if $6s+7-l_1 = 8s+9-l_2$ then $l_1 \cdot l_2 = \{10s+11-l_1\}$ and $l_2 \cdot l_1 = \{l_1, l_1+1, \dots, 14s+15-l_2(12s+13-l_1)\}$. If $6s+7-l_1 < 8s+9-l_2$

then $l_1 \cdot l_2 = \{l_2 - (2s+2)\}$ and $l_2 \cdot l_1 = \{14s+15-l_2\}$. If $6s+7-l_1 > 8s+9-l_2$ then $l_1 \cdot l_2 = \{l_1\}$ and $l_2 \cdot l_1 = \{l_1 - (2s+2)\}$.

Let now $l_2 = 8s+8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$.

Case 12. $l_1 = 6s+6$.

We have: $K_0(r^2(x), t, r^3(x)) \wedge E_1^*(t, r^3(x))$. Let also $l_2 = 6s+6$, i.e. $K_0(r^2(t), y, r^3(t)) \wedge E_1^*(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{4s+4, 4s+5\}$.

Let now $l_2 = 6s+7$, i.e. $K_0(r^3(t), y, t) \wedge E_1^*(y, r^3(t))$.

We have the following: $l_1 \cdot l_2 = \{4s+5\}$ and $l_2 \cdot l_1 = \{4s+4\}$.

Let now $6s+8 \leq l_2 \leq 7s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{l_2-(6s+6)}^*(y, r^3(t)) \wedge \neg E_{l_2-(6s+7)}^*(y, r^3(t)).$$

We have the following: $l_1 \cdot l_2 = \{l_2 - (2s+2)\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$.

Let now $7s+8 \leq l_2 \leq 8s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}(y, t) \wedge \neg E_{8s+8-l_2}(y, t).$$

We have the following: $l_1 \cdot l_2 = \{l_2 - (2s+2)\}$ and $l_2 \cdot l_1 = \{14s+15-l_2\}$.

Let now $l_2 = 8s+8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = \{6s+6\}$ and $l_2 \cdot l_1 = \{6s+6, 6s+7\}$.

Case 13. $l_1 = 6s+7$.

We have: $K_0(r^3(x), t, x) \wedge E_1^*(t, r^3(x))$. Let also $l_2 = 6s+7$,

i.e. $K_0(r^3(t), y, t) \wedge E_1^*(y, r^3(t))$. Then we have $l_1 \cdot l_2 = \{4s+4, 4s+5\}$.

Let now $6s+8 \leq l_2 \leq 7s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{l_2-(6s+6)}^*(y, r^3(t)) \wedge \neg E_{l_2-(6s+7)}^*(y, r^3(t)).$$

We have the following: $l_1 \cdot l_2 = \{l_2 - (2s+2)\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$.

Let now $7s+8 \leq l_2 \leq 8s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}(y, t) \wedge \neg E_{8s+8-l_2}(y, t).$$

We have the following: $l_1 \cdot l_2 = \{l_2 - (2s+2)\}$ and $l_2 \cdot l_1 = \{14s+15-l_2\}$.

Let now $l_2 = 8s+8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = \{6s+6, 6s+7\}$ and $l_2 \cdot l_1 = \{6s+7\}$.

Case 14. $6s+8 \leq l_1 \leq 7s+7$.

We have: $K_0(r^3(x), t, x) \wedge E_{l_1-(6s+6)}^*(t, r^3(x)) \wedge \neg E_{l_1-(6s+7)}^*(t, r^3(x))$.

Let also $6s+8 \leq l_2 \leq 7s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{l_2-(6s+6)}^*(y, r^3(t)) \wedge \neg E_{l_2-(6s+7)}^*(y, r^3(t)).$$

We have the following: if $l_1 = l_2$ then $l_1 \cdot l_2 = \{10s+11-l_1, \dots, l_1-(2s+2)\}$.

If $l_1 < l_2$ then $l_1 \cdot l_2 = \{l_2 - (2s+2)\}$ and $l_2 \cdot l_1 = \{10s+11-l_2\}$.

If $l_1 > l_2$ then $l_1 \cdot l_2 = \{10s+11-l_2\}$ and $l_2 \cdot l_1 = \{l_2 - (2s+2)\}$.

Let now $7s+8 \leq l_2 \leq 8s+7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}(y, t) \wedge \neg E_{8s+8-l_2}(y, t).$$

We have the following: if $l_1 - (6s+6) = 8s+9-l_2$ then $l_1 \cdot l_2 = \{12s+13-l_1, \dots, l_1\}$ and $l_2 \cdot l_1 = \{14s+15-l_2\}$. If $l_1 - (6s+6) < 8s+9-l_2$ then

$l_1 \cdot l_2 = \{l_2 - (2s + 2)\}$ and $l_2 \cdot l_1 = \{14s + 15 - l_2\}$. If $l_1 - (6s + 6) > 8s + 9 - l_2$ then $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$.

Let now $l_2 = 8s + 8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = \{10s + 11 - l_1\}$ and $l_2 \cdot l_1 = \{l_1\}$.

Case 15. $7s + 8 \leq l_1 \leq 8s + 7$.

We have: $K_0(r^3(x), t, x) \wedge E_{8s+9-l_1}(t, x) \wedge \neg E_{8s+8-l_1}(t, x)$.

Let also $7s + 8 \leq l_2 \leq 8s + 7$, i.e.

$$K_0(r^3(t), y, t) \wedge E_{8s+9-l_2}(y, t) \wedge \neg E_{8s+8-l_2}(y, t).$$

We have the following: if $l_1 = l_2$ then $l_1 \cdot l_2 = \{l_1\}$. If $l_1 < l_2$ then $l_1 \cdot l_2 = \{l_2\}$. If $l_1 > l_2$ then $l_1 \cdot l_2 = \{l_1\}$.

Let now $l_2 = 8s + 8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = l_2 \cdot l_1 = \{l_1\}$.

Case 16. $l_1 = 8s + 8$.

We have: $K_0(r^3(x), t, x) \wedge E_1(t, x)$. Let also $l_2 = 8s + 8$, i.e. $K_0(r^3(t), y, t) \wedge E_1(y, t)$. Then we have $l_1 \cdot l_2 = \{l_1\}$.

Thus, we established that the algebra $\mathfrak{P}_{M'_{s,m,k}}$ is non-commutative and strictly $(2s + 3)$ -deterministic for all valid values s, m and k . $\square$

3 Conclusion

We investigated algebras of binary isolating formulas for $\aleph_0$ -categorical 1-transitive non-primitive weakly circularly minimal theories of convexity rank greater than 1 with a trivial definable closure having a non-trivial piecewise monotonic-to-left function acting on the universe of a structure with an arbitrary s and $m = k = 4$. We also proved their non-commutativity and established their strict l -deterministicity for some natural l . It would now be interesting to describe the corresponding algebras for such theories having arbitrary s, m and k .

References

- [1] A.B. Altayeva, B.Sh. Kulpeshov, S.V. Sudoplatov, *Algebras of distributions of binary isolating formulas for almost ω -categorical weakly o-minimal theories*, Algebra and Logic, **60**:4 (2021), 241–262.
- [2] A.B. Altayeva, B.Sh. Kulpeshov, *Almost binarity of countably categorical weakly circularly minimal structures*, Mathematical Notes, **110**:6 (2021), 813–829.
- [3] M. Bhattacharjee, H.D. Macpherson, R.G. Möller, P.M. Neumann, *Notes on Infinite Permutation Groups*, Lecture Notes in Mathematics 1698, Springer, 1998. Zbl 0916.20002
- [4] P.J. Cameron, *Orbits of permutation groups on unordered sets, II*, J. Lond. Math. Soc., II. Ser., **23** (1981), 249–264. Zbl 0498.20003
- [5] M. Droste, M. Giraudet, H.D. Macpherson and N. Sauer, *Set-homogeneous graphs*, J. Comb. Theory, Ser. B, **62**:1 (1994), 63–95.

- [6] D.Yu. Emelyanov, *On algebras of distributions of binary formulas for theories of unars*, Bulletin of Irkutsk State University. Series Mathematics, **17** (2016), 23–36. Zbl 1362.03028
- [7] D.Yu. Emelyanov, *Algebras of binary isolating formulas for theories of Cartesian products of graphs*, Algebra and Model Theory 12: Coll. of papers, Novosibirsk: NSTU, 2019, 21–31.
- [8] D.Yu. Emelyanov, *Algebras of binary isolating formulas*, Algebra and Logic, **60**:4 (2021), 288–291.
- [9] D.Yu. Emelyanov, *Algebras of binary isolating formulas for strong product theories*, Bulletin of Irkutsk State University. Series Mathematics, **45** (2023), 138–144.
- [10] D.Yu. Emelyanov, *Algebras of binary isolating formulas for graphs with simplexes*, Algebra and Model Theory 14: Coll. of papers, Novosibirsk: NSTU, 2023, 41–44.
- [11] D.Yu. Emelyanov, *Algebras of binary isolating formulas for Cartesian products of graphs*, Model Theory and Algebra 2024: Coll. of papers, Novosibirsk: NSTU, 2024, 25–31.
- [12] B.Sh. Kulpeshov, S.V. Sudoplatov, *Algebras of binary formulas for weakly circularly minimal theories with non-trivial definable closure*, Lobachevskii Journal of Mathematics, **43**:12 (2022), 3532–3540.
- [13] B.Sh. Kulpeshov, *Algebras of binary formulas for $\aleph_0$ -categorical weakly circularly minimal theories: piecewise monotonic case*, Siberian Electronic Mathematical Reports, **20**:2 (2023), 824–832.
- [14] B.Sh. Kulpeshov, S.V. Sudoplatov, *Algebras of binary formulas for $\aleph_0$ -categorical weakly circularly minimal theories: monotonic case*, Bulletin of the Karaganda University. Mathematics series, **1** (113), 2024, 112–127.
- [15] B.Sh. Kulpeshov, *Algebras of binary formulas for weakly circularly minimal theories with trivial definable closure*, Sib. Math. J., **65**:2 (2024), 316–327.
- [16] B.Sh. Kulpeshov, H.D. Macpherson, *Minimality conditions on circularly ordered structures*, Math. Logic Quart., **51**:4 (2005), 377–399.
- [17] B.Sh. Kulpeshov, *On $\aleph_0$ -categorical weakly circularly minimal structures*, Math. Logic Quart., **52**:6 (2006), 555–574.
- [18] B.Sh. Kulpeshov, *Definable functions in the $\aleph_0$ -categorical weakly circularly minimal structures*, Sib. Math. J., **50**:2 (2009), 282–301.
- [19] B.Sh. Kulpeshov, *Weakly o-minimal structures and some of their properties*, J. Symb. Log., **63**:4 (1998), 1511–1528.
- [20] B.Sh. Kulpeshov, *A criterion for binarity of almost ω -categorical weakly o-minimal theories*, Sib. Math. J., **62**:2 (2021), 1063–1075.
- [21] H.D. Macpherson, D. Marker, and C. Steinhorn, *Weakly o-minimal structures and real closed fields*, Trans. Amer. Math. Soc., **352**:12 (2000), 5435–5483.
- [22] I.V. Shulepov, S.V. Sudoplatov, *Algebras of distributions for isolating formulas of a complete theory*, Sib. Electron. Math. Rep., **11** (2014), 380–407.
- [23] S.V. Sudoplatov, *Classification of countable models of complete theories*. Novosibirsk: NSTU, 2018 (in Russian).

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