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NOTES ON SOME VERSIONS OF CO-HOPFIAN ABELIAN GROUPS

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Abstract: We establish some additional results concerning relatively co-Hopfian Abelian groups, recently defined by us in the Siber. Math. J. (2026). Concretely, for any prime p, we prove that relatively co-Hopfian p-groups are thick as well as that separable relatively co-Hopfian p-groups are thin. Moreover, we show that a reduced fully-starred p-group is relatively co-Hopfian exactly when it is finite. Finally, we demonstrate that relatively co-Hopfian groups are closed under the formation of finitely generated extension.

Keywords: Abelian groups, directly finite groups, co-Hopfian groups, co-Bassian groups, generalized co-Bassian groups.

1 Introduction and Background

Throughout this article, all groups into consideration are *additively* written and *Abelian*. Our basic notation and terminology will standardly follow those from the books [4, 5]. As usual, for some prime integer p, $\mathbb{Z}(p^n)$ denotes the cyclic p-group of order p^n for some $n \geq 1$, and $\mathbb{Z}(p^\infty)$ stands for the quasi-cyclic divisible p-group.

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To Brendan Goldsmith on his **75***th birthday.*

A classical concept of some majority in certain directions of Abelian group theory is the one of a *co-Hopfian* group G which states that any monomorphism (in other terms, an injective endomorphism) $\phi : G \to G$ is surjective, i.e., $\phi(G) = G$ (or, equivalently, ϕ is an bijective homomorphism, that is, an isomorphism). So, $\phi(G)$ is a trivial (i.e., a non-proper) direct summand of G. In other words, restating this in an equivalent manner, G is a group that is not isomorphic to any of its proper subgroups.

This restatement motivated us in [2] to formulate and study in-depth the following two notions.

• We shall say that a group G is *relatively co-Hopfian* if, for every monomorphism, $\phi : G \to G$, the image $\phi(G)$ is essential in G.

• We shall say that a group G is *generalized co-Hopfian* if, whenever $\phi : G \to G$ is a monomorphism, then $\phi(G)$ is a direct summand of G.

In what follows, we are planning to discover some extra substantial results on these two classes of groups, thus significantly supplying our results from [2]. Specifically, we are succeeding in obtaining the following achievements: for any prime p, relatively co-Hopfian p-groups are thick (see Proposition 3). Likewise, we show that separable relatively co-Hopfian p-groups are thin (see Proposition 4). Also, we prove that reduced fully-starred relatively co-Hopfian p-groups are finite (see Proposition 5). Besides, we establish that the property of being relatively co-Hopfian is inherited under the formation of its direct sum with an arbitrary finite generation (see Proposition 6).

2 Results and Examples

Standardly, as noticed above, for the maximal torsion subgroup $T(G) = \oplus_p T_p(G)$ of a group G, the symbol $T_p(G)$ stands for its p-torsion component for some prime p.

The next statement is somewhat helpful and close to the preceding assertion, so it is worthy of documentation (at least that it seems to be unpublished in the existing literature).

Proposition 1. *The subgroup p^nG is essential in G for every integer $n > 0$ if, and only if, $T_p(G)$ is divisible.*

Proof. **Necessity.** If, in a way of contradiction, we assume that $T_p(G) \neq \{0\}$ is not divisible, then $T_p(G)$ contains a non-zero cyclic direct summand, say H, of G of order $k > 0$, and hence $H \cap p^{k+1}G = \{0\}$, as pursued.

Sufficiency. Letting H be a non-zero cyclic subgroup in G, if $H \cap T_p(G) \neq \{0\}$, then $H \cap p^nG \neq \{0\}$ for every integer $n > 0$. Furthermore, if $H \cap T_p(G) = \{0\}$, then it is not so difficult to verify that it is possible to choose the subgroup B of G such that $G = T_p(G) \oplus B$ and $H \leq B$ (see, e.g., [4, Theorem 21.2]). But, since $T_p(B) = \{0\}$, we deduce that $\{0\} \neq p^nH \leq p^nG$ for every $n > 0$, as asked for. □

Before proceeding by proving our next result on direct sums, we need one more technicality like this.

Lemma 1. *Let $G = A \oplus B$ with projection $\pi : G \to A$, and let $H = K \oplus N \oplus F \leq G$, where N is an essential subgroup of B and the subgroup $\pi(K)$ is essential in A. Then, $F = \{0\}$.*

Proof. Since N is essential in B, we have $F \cap B = \{0\}$. We now intend to prove that $F \cap A = \{0\}$, so assume the contrary that $0 \neq x \in F \cap A$. Since $\pi(K)$ is essential in A, there exists $n \in \mathbb{N}$ such that $nx \neq 0$ and $nx + b \in K$ for some $b \in B$. Note that $b \neq 0$, as for otherwise $0 \neq nx \in F \cap K = \{0\}$. If now $0 \neq mb \in N$ for $m \in \mathbb{N}$, then

$$nmx = (nmx + mb) - mb \in (K \oplus N) \cap F = \{0\}.$$

Therefore, $nmx = 0$ and hence

$$0 \neq mb = nmx + mb \in K \cap N = \{0\},$$

a contradiction. So, it must be that $F \cap A = \{0\}$.

Assume now that $F \neq \{0\}$ and $0 \neq x \in F$. Then, $x = a + b$, where, in view of the already obtained above equalities $F \cap A = F \cap B = \{0\}$, we have $0 \neq a \in A$, $0 \neq b \in B$. Thus, there exists $m \in \mathbb{N}$ such that $ma \neq 0$ and $y := ma + b' \in K$ for some $b' \in B$. Hence, $mx - y = mb - b' \in B$, where $mx - y \neq 0$ since $mx \neq 0$, $y \neq 0$ and $K \cap F = \{0\}$. Consequently, $b_1 := mb - b' \neq 0$, where $b_1 \in B$. Observe also that there exists $n \in \mathbb{N}$ such that $0 \neq nb_1 \in N$. Thus,

$$0 \neq nb_1 = nmx - ny \in (F \oplus K) \cap N = \{0\}.$$

This contradiction gives that $F = \{0\}$, as claimed. □

It was proved in [2, Lemma 2.1] that a direct summand of a relatively co-Hopfian group is too relatively co-Hopfian. Moreover, it was proven in [2, Proposition 2.6] that the direct sum of two relatively co-Hopfian groups, one of which is fully invariant in this sum, is again relatively co-Hopfian.

We now have enough instruments to attack the truthfulness of this claim by using a more transparent approach.

Proposition 2. *Let $G = A \oplus B$, where B is fully invariant in G and both A, B are relatively co-Hopfian groups. Then, G also is a relatively co-Hopfian group.*

Proof. Write

$$H = K \oplus N \oplus F \leq G,$$

where $K \cong A$ and $N \cong B$. Since B is fully invariant in G, we get $N \leq B$, and since $N \cong B$, we have that N is essential in B by referring to the corresponding definition listed above. So, $K \cap B = \{0\}$ whence, if $\pi : G \to A$ is a projection, then having in mind the isomorphisms

$$\pi(K) \cong K \cong A,$$

one verifies that the subgroup $\pi(K)$ is essential in A by the same definition. Consequently, Lemma 1 tells us that $F = \{0\}$. So, the group G is really relatively co-Hopfian, as asserted. □

Recall that a p-group G is said to be *thick*, if whenever B is a direct sum of cyclic p-groups and $\phi : G \to B$ is a homomorphism, then ϕ is *small*, i.e., the kernel of ϕ is a *large* subgroup of G (for a more detailed information, we refer to [4, Section 67]).

To use this notion in the present context, we recall the following helpful result:

• ([7, Corollary 18(a)]) *If G is a p-group that is not thick, then there is a p-group H with a direct summand that is an unbounded direct sum of cyclic groups such that G and H embed in each other.*

We, thereby, come to the following more concrete statement.

Proposition 3. *If the p-group G is relatively co-Hopfian, then it is thick.*

Proof. Suppose the contrary that G is *not* thick and $H = A \oplus B$, where B is an unbounded direct sum of cyclic groups such that G and H embed in each other. Since B, and hence H, is *not* relatively co-Hopfian, it follows that H embeds into itself as a non-essential subgroup. Therefore, G also embeds in H as a non-essential subgroup. But since H embeds in G, one has that G must embed in itself as a non-essential subgroup. Consequently, G is *not* relatively co-Hopfian, as expected. □

Note that an unbounded torsion-complete p-group will be thick, but not co-Hopfian, so the converse to the last result does *not* generally hold. It, however, does have in what follows an interesting consequence.

Recall that the so-called $\oplus_c$-*topology* uses the subgroups $X \leq G$ such that G/X is a direct sum of cyclic groups as a neighborhood base of $\{0\}$ (see, for a more account, [3]).

Corollary 1. *Suppose G is a relatively co-Hopfian p-group. Then, G is complete in its $\oplus_c$-topology if, and only if, it is finite.*

Proof. Sufficiency being pretty obvious, we now focus on the necessity. To that end, suppose now that G is complete in its $\oplus_c$-topology; in particular, G must be separable. Since G is thick, one knows that its $\oplus_c$-completion agrees with its torsion-completion, say $\overline{G}$ (see cf. [3, Proposition 1.1]). So, [2, Lemma 2.20] works successfully to derive that $G \cong \overline{G}$ must be finite, as stated. □

We now mention a specific case of the last above result. In [5], L. Fuchs designated the smallest class of Abelian p-groups containing the cyclic groups that is closed with respect to direct sums, direct summands, and the torsion subgroups of direct products over non-measurable index sets as the *Keef class*, denoting it by K_p. Clearly, K_p also contains both the direct sums of cyclic p-groups and the torsion complete p-groups. It is principally known that the elements of K_p are all complete in their $\oplus_c$-topologies.

Therefore, the following statement is an immediate consequence of Corollary 1.

Corollary 2. *A group $G \in K_p$ is relatively co-Hopfian if, and only if, it is finite.*

We now consider a property that is, in some sense, dual to that of being thick. The p-group G is said to be *thin* if, for every torsion-complete p-group $\overline{B}$, any homomorphism $\phi : \overline{B} \to G$ must be small. In this vein, Megibben showed that, if G is separable, then G is thin if, and only if, it does not have a subgroup that is isomorphic to an unbounded torsion-complete p-group ([10, Theorem 3.1]). Thus, the following relates this to our inquiry.

Proposition 4. *If G is a separable p-group that is relatively co-Hopfian, then G is thin.*

Proof. We suppose the contrary that G is a separable p-group that is not thin, and show that G is also not relatively co-Hopfian. By the aforementioned result from [10], G must have a subgroup $H \cong \overline{B}$, where B is an unbounded direct sum of cyclic groups. Utilizing now a familiar argument, $\overline{B}$ has an unbounded proper direct summand, and replacing H with this summand, there is no loss of generality in assuming that H is not essential in G.

Note that since the group G is semi-standard (that is, all its Ulm-Kaplansky invariants defined over the set of finite ordinals always have finite value), if A is a basic subgroup of G, then A is also semi-standard, and hence the direct sum of a countable number of cyclic direct summands. By embedding cyclic direct summands of A into larger cyclic direct summands of B, there is a monomorphism $\gamma : A \to B$ which extends to a monomorphism $\overline{A} \to \overline{B}$. Since G embeds in $\overline{A}$ which embeds in $\overline{B}$ which embeds as a non-essential subgroup of G again, we can conclude that G is not relatively co-Hopfian, as stated. □

It follows from a combination of Propositions 3 and 4 that a separable relatively co-Hopfian p-group is *thick-thin*, a class of groups that received considerable attention in [9].

Recall that the reduced p-group G is said to be *fully starred* if every subgroup of G has the same cardinality as one of its basic subgroups. For example, if H and K are reduced p-groups, it is known that the *torsion functor* $\mathrm{Tor}(H, K)$ between H and K (and all of its subgroups) will be fully starred (see, for example, cf. [4, Chapter X]). In particular, this implies that, for any ordinal α, any subgroup of a p^α-pure projective p-group will be fully starred. In addition, a countable reduced p-group is fully starred.

We can now record the following.

Proposition 5. *If G is a reduced fully-starred p-group, then G is relatively co-Hopfian if, and only if, it is finite.*

Proof. Certainly, if G is finite, then it is co-Hopfian and thus relatively co-Hopfian.

So, assume G is an infinite reduced and fully-starred relatively co-Hopfian p-group. Since G must be semi-standard, if it is bounded, then it must be finite. Reciprocally, if it is assumed on the contrary to be unbounded, then it must have a countably infinite basic subgroup, so that G is also simultaneously countably infinite and unbounded. Therefore, $G/p^\omega G$ is too countably infinite and unbounded, and hence an unbounded direct sum of cyclic groups being separable (see, for instance, [4, Chapter III, Section 17] or [5, Chapter 3]). But, an appeal to [2, Proposition 2.18] reaches that the group G is co-Hopfian, and thus [1] enables us that the factor-group $G/p^\omega G$ is co-Hopfian as well, and so Proposition 3 allows us to infer that the group G, and hence the quotient $G/p^\omega G$, must be thick, giving the desired contradiction. □

The following result on co-Hopficity is, principally, well-documented (see, e.g., [6]).

Corollary 3. *A countable reduced p-group is relatively co-Hopfian if, and only if, it is finite.*

Consulting with [2, Proposition 2.23], we may derive the following helpful consequence.

Corollary 4. *Suppose G is a group with torsion T. If T is co-Hopfian and G/T has finite rank, then G is relatively co-Hopfian.*

The following comments could somewhat be useful. They illustrate that the restrictions used by us in our affirmations are the most minimal possible.

Remark 1. *It is worthwhile noticing that, exploiting an analogous idea, the cited assertion from* [2] *can be extended thus: If F is a fully invariant subgroup of a group G such that F is co-Hopfian and G/F is relatively co-Hopfian, then G itself is relatively co-Hopfian.*

It is also worthy of noticing that, for every prime p, let $Q_p \subseteq \mathbb{Q}$ be the group consisting of the fractions of the form a/p^k such that $a \in \mathbb{Z}$, $k \in \mathbb{N}$. In particular, for all $p \in \mathcal{P}$, the group Q_p has type greater than $\mathbb{Z}$ and there are an infinite number of such Q_p. Considering now the group

$$G := \mathbb{Z} \oplus (\bigoplus_{p \in \mathcal{P}} Q_p),$$

we, however, discover that this group G pretty clearly is relatively co-Hopfian. This expectedly satisfies [2, Proposition 2.26] *in which proposition we require the condition for non-existence of an infinite subset $\{i_k \ : \ k \in \mathbb{N}\}$ such that $\tau_k < \tau_{k+1}$ for each $k \in \mathbb{N}$. This is, however, in a sharp contrast with the requirement that "each $t \in \Omega$ has only finite type greater than t" and, that is why, a claim of the sort "The completely decomposable torsion-free group G for which $G = \bigoplus_{t \in \Omega} G_t$, where all direct summands G_t are homogeneous components of G and Ω is a set of same type direct summands of G of rank 1, is relatively co-Hopfian if, and only if, each G_t has finite rank and each $t \in \Omega$ has only finite type greater than t" cannot happen as the constructed above group G unambiguously shows (compare with* [8] *as well).*

We are now concerned with the inheritance of the direct sum property via the relative co-Hopficity, intending to confirm the following two statements, the first of which generalizes significantly the corresponding well-known result for co-Hopficity.

Proposition 6. *If G is a relatively co-Hopfian group and F is a finitely generated group, then $G \oplus F$ too is a relatively co-Hopfian group.*

Proof. One sees that it is enough to prove the assertion when $F = \langle a \rangle$ is a cyclic group of either infinite order or an order p^n for some prime number p and an integer $n > 0$.

To this target, suppose that $A = G \oplus \langle a \rangle$ and $\pi : A \to G$ is the corresponding projection. Assume that $H = C \oplus \langle b \rangle \oplus X \subsetneqq A$, where $C \cong G$, order(b) = order(a) and $X \neq \{0\}$.

Let order$(a) = p^n$. Consider two basic cases as follows:

Case 1: Write $b = g + a$ for some $g \in G$ (accurate to certain integer multiple, mutually simple with p). We have $A = G \oplus \langle a + g \rangle$ and so $H = (H \cap G) \oplus \langle a + g \rangle$. Here, $H \cap G \cong C \oplus X$ and $H \cap G \subsetneqq G$ since relatively co-Hopfian groups do not have proper direct summands isomorphic to itself. However, as $C \cong G$ and $X \neq \{0\}$, which is impossible for a relatively co-Hopfian group.

Case 2: Write $b = g + p^l a$, where $0 < l \leq n$. Thus, order(b) = order(g). Since $p^{n-l} b = p^{n-l} g \neq 0$, it must be that $\langle g \rangle \cap (C \oplus X) = \{0\}$. We, thus, arrive at the subgroup $H' = C \oplus \langle g \rangle \oplus X \leq A$, which obviously is a proper direct decomposition.

If $C \cap \langle a \rangle = \{0\}$, then $\pi(C) \oplus \pi(\langle g \rangle) \subsetneqq G$, where $\pi(C) \cong C \cong G$ and $\pi(\langle g \rangle) = \langle g \rangle$, which does not hold for a relatively co-Hopfian group.

Assume now that $C \cap \langle a \rangle \neq \{0\}$, so that $c = p^t a \in C$ for some $0 \leq t < n$. Therefore, $C \cap \langle a + g \rangle = \{0\}$. In fact, if $c_1 = p^s(a + g) \in C$ for some $0 \leq s < n$, then, for $m = \max\{t, s\}$, we have

$$0 \neq p^{m-s} c_1 - p^m a = p^m g \in \langle g \rangle \cap C,$$

a contradiction, as expected.

Next, if $\pi' : A \to G$ is the corresponding projection of a group $A = G \oplus \langle a + g \rangle$ on G, then

$$\pi'(C \oplus \langle g \rangle) = \pi'(C) \oplus \langle g \rangle \subsetneqq G,$$

where $\pi'(C) \cong G$, which is manifestly wrong for a relatively co-Hopfian group.

Finally, let order$(a) = \infty$. Assume $b = g + ka$ for some integer $k \neq 0$. So,

$$H = (H \cap G) \oplus \langle g + ka \rangle \leq A' = G \oplus \langle g + ka \rangle,$$

where $H \cap G \cong C \oplus X$ and $H \cap G \subsetneqq G$, which is false for a relatively co-Hopfian group.

Let us now we have $b \in G$. If $C \cap \langle a \rangle = \{0\}$, then $\pi(C) \oplus \langle b \rangle \subsetneqq G$, where $\pi(C) \cong G$, which is untrue for a relatively co-Hopfian group. But, if $C \cap \langle a \rangle \neq \{0\}$, then $c = ka \in C$ for some integer $k \neq 0$. Thus, for $A = G \oplus \langle a + b \rangle$, we obtain $C \cap \langle a + b \rangle = \{0\}$, because if $c_1 = k(a + b)$, then

$$nc_1 - nka = nkb \in \langle b \rangle \cap C = \{0\},$$

a contradiction. So, if $\pi'' : A \to G$ is a projection of $A = G \oplus \langle a + b \rangle$ on G, then argued as above $\pi''(C) \oplus \langle b \rangle \subsetneqq G$ and $\pi''(C) \cong G$, which is manifestly *not* fulfilled for a relatively co-Hopfian group. □

Recall that a groups is termed *finitely co-generated*, provided that its reduced part and divisible part have a respective finite generating set.

As a valuable consequence, we have:

Corollary 5. *If G is a relatively co-Hopfian group and F is a finitely co-generated group, then $G \oplus F$ too is a relatively co-Hopfian group.*

Proof. Since any finitely co-generated group is known to be a finite direct sum of co-cyclic groups (i.e., a direct sum of a finite number of cyclic and divisible groups), then the result follows from a combination between Proposition 6 and Proposition 2, equipped with the construction of a divisible relatively co-Hopfian group given in [2]. □

Further, one may expect that an analogous claim to that in Proposition 6 will hold for generalized co-Hopfian groups, namely that if G is a generalized co-Hopfian group and F is a finitely generated group, then $G \oplus F$ is too a generalized co-Hopfian group, but unfortunately this is *not* manifestly invalid. In fact, as shown in [2], any torsion-free generalized co-Hopfian group is divisible, and even if F is taken to be torsion (and hence finite), [2, Theorem 2.31] will teach us that $f_k(F) = 0$ for all $k < n$, which means that F is semi-standard, that is manifestly untrue.

3 Concluding Discussion and Open Questions

In order to provide the interested reader with some more useful information, we intend here to comment some things in other light.

They are closely connected to the procedure described in [1, Section 4], where it is demonstrated that if a p-group G is co-Hopfian, then so does its first Ulm factor $G/p^\omega G$, as well as some other important things under the presence of the Martin's axiom (MA). We thus arrive at the following two challenging questions, the first one of which is pertained to [2, Proposition 2.4] and the second one to [2, Theorem 2.31], but both of them are results in (ZFC) without any extra set-theoretic assumptions.

Problem 1. If a p-group G is directly finite, is its first Ulm factor $G/p^\omega G$ also directly finite?

Solution. The question follows directly from the cited above Proposition 2.4 from [2], because it must be that

$$f_n(G/p^\omega G) = f_n(G)$$

for $n < \omega$ and

$$f_\alpha(G/p^\omega G) = 0$$

for $\alpha \geq \omega$ (see [4, Chapter XII, Section 77]). This terminates our proof.

Note that, if a p-group G is relatively co-Hopfian, then its first Ulm factor $G/p^\omega G$ is also relatively co-Hopfian. In fact, just a simple combination of the conclusions in [1, Section 4] and [2, Proposition 2.18] works to get the claim.

Problem 2. If a p-group G is generalized co-Hopfian, is its first Ulm factor $G/p^\omega G$ also generalized co-Hopfian?

Solution. Thankfully, [2, Theorem 2.31] yields that $G = A \oplus C$, where C is a p^n-bounded group and A is a co-Hopfian group. So, one writes that

$$G/p^\omega G \cong (A/p^\omega A) \oplus C.$$

Furthermore, since by what we have commented above $A/p^\omega A$ is co-Hopfian, and

$$f_m(A/p^\omega A) = f_m(A)$$

for every $m < \omega$ (see [4, Chapter XII, Section 77]), again the same [2, Theorem 2.31] is workable to get that $G/p^\omega G$ is generalized co-Hopfian, as pursued. The proof is over.

So, we come to our final question which settling is unknown to us yet.

Problem 3. Find suitable conditions on the first Ulm subgroup $p^\omega G$ of an arbitrary p-group G under which, if the first Ulm factor $G/p^\omega G$ is directly finite (resp., generalized co-Hopfian), then so does the former group G.

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