S©MR

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Vol. 22, No. 2, pp. 1731–1740 (2025) УДК 519.214
https://doi.org/10.33048/semi.2025.22.105 MSC 60F15

STRASSEN'S LAW OF ITERATED LOGARITHM FOR A WIENER PROCESS DEFINED ON THE HALF-AXIS

E.V. EFREMOV AND **A.V. LOGACHOV**

Communicated by N.S. Arkashov

Abstract: We prove a version of Strassen's law of iterated logarithm in the space of continuous functions with weighted sup−metric for a Wiener process defined on the half-axis. We consider weight functions of the form $1/(1+t^\alpha)$, where $\alpha > 1/2$. The result is unimprovable in the class of power weight functions.

Keywords: Wiener process, Strassen's law of iterated logarithm.

1 Introduction, formulation of the main result

Let $w(t)$, $t \geqslant 0$ be a Wiener process defined on a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$. We will be interested in the limit behavior of the following sequence of random processes

$$w_n(t) = \frac{w(nt)}{\sqrt{n}\varphi(n)}, \quad n \in \mathbb{N},$$

where $\varphi(n) := \sqrt{2 \ln\ln(3 \vee n)}$. Note that for any $n \in \mathbb{N}$ the random process $\tilde{w}(t) = w(nt)/\sqrt{n}$ is also a Wiener process.

Efremov, E.V., Logachov, A.V., Strassen's law of iterated logarithm for a Wiener process defined on the half-axis.

The work of E.V. Efremov is supported by the Mathematical Center in Akademgorodok under the agreement No. 075-15-2025-348 with the Ministry of Science and Higher Education of the Russian Federation. A.V. Logachov thanks the Basic Research Program of Siberian Branch of Russian Academy of Sciences, project FWNF-2022-0010.

Received September, 23, 2025, Published December, 31, 2025.

We will consider trajectories of random processes w_n in the space $\mathbb{C}$ of functions continuous on the half-axis $[0,\infty)$, with a metric

$$\rho(f,g)=\rho_\alpha(f,g):=\sup_{t\geqslant 0}\frac{|f(t)-g(t)|}{1+t^\alpha},$$

where $1/(1+t^\alpha)$ is a power weight function and $\alpha>\frac{1}{2}$ is a fixed constant.

Let us denote as $\mathbb{AC}_0[0,\infty)$ and $\mathbb{AC}_0[0,T]$ sets of functions starting from zero and absolutely continuous on intervals $[0,\infty)$ and $[0,T]$ respectively.

We define the Strassen's ball on the half-axis

$$K:=\left\{f\in\mathbb{AC}_0[0,\infty):\int_0^\infty \dot f^2(s)ds\leqslant 1\right\},$$

where $\dot f$ is the derivative in the sense of absolute continuity.

We also define the Strassen's ball on $[0,T]$

$$K_T:=\left\{f\in\mathbb{AC}_0[0,T]:\int_0^T \dot f^2(s)ds\leqslant 1\right\}.$$

The following Theorem is the main result of the paper.

Theorem 1. *For any $\alpha>\frac{1}{2}$ the set limit points of the sequence w_n in the metric space $\mathbb{C}$ coincides with the set K a.s.*

Now we will do a brief review of results obtained earlier. The case of $\alpha=1$ was considered in the paper [2, Theorem 1.4.1] for multidimensional Wiener process; the result then was used to prove the Strassen's law of iterated logarithm for small time in [3, Theorem 3]. The paper [4, Theorem 1] is also dedicated to the case, when $\alpha=1$, but a class of more general normalizing functions than $\varphi(n)=\sqrt{2\ln\ln(3\vee n)}$ was considered. We note that the methods of proof used in this paper differ significantly form those proposed in [4]. We also mention the paper [5]. In that paper the moderate deviations principle for diffusion processes was obtained for the case when $\alpha\geqslant 1$, which is the main tool for proving the Strassen's law of iterated logarithm.

Remark 1. *In the case when $\alpha\leqslant\frac{1}{2}$, the Strassen's law of iterated logarithm cannot be obtained, since for such α, due to the Khinchin's law of iterated logarithm, the following holds*

$$\limsup_{t\to\infty}\frac{|\tilde w(t)|}{1+t^\alpha}\geqslant\limsup_{t\to\infty}\frac{|\tilde w(t)|}{1+\sqrt{t}}=\infty\quad a.s. \tag{1}$$

Also, due to the Cauchy–Bunyakovsky–Schwarz inequality, for any $f\in K$ the following bound holds

$$|f(t)|\leqslant\int_0^t|\dot f(s)|ds\leqslant\sqrt{t}\left(\int_0^t\dot f^2(s)ds\right)^{1/2}\leqslant\sqrt{t}. \tag{2}$$

Using formulas (1) and (2) we can conclude that for any $\alpha \leqslant \frac{1}{2}$, any function $f \in K$ and any $n \in \mathbb{N}$ the following holds

$$\begin{aligned}\rho_\alpha(w_n, f) &= \sup_{t \geqslant 0} \frac{|w_n(t) - f(t)|}{1+t^\alpha} \geqslant \sup_{t \geqslant 0} \left(\frac{|w_n(t)|}{1+t^\alpha} - \frac{|f(t)|}{1+t^\alpha} \right) \\ &\geqslant \sup_{t \geqslant 0} \left(\frac{|w_n(t)|}{1+t^\alpha} - \frac{\sqrt{t}}{1+t^\alpha} \right) = \sup_{t \geqslant 0} \frac{1+\sqrt{t}}{1+t^\alpha} \left(\frac{|w_n(t)|}{1+\sqrt{t}} - \frac{\sqrt{t}}{1+\sqrt{t}} \right) \\ &\geqslant \sup_{t \geqslant 0} \frac{1+\sqrt{t}}{1+t^\alpha} \left(\frac{|w_n(t)|}{1+\sqrt{t}} - 1 \right) = \infty \quad \text{a.s.}\end{aligned}$$

Thus, the result of Theorem 1 is unimprovable in the class of power weight functions.

Let us introduce some more new notations. For fixed $T \in (0, \infty)$ we will denote the space of functions continuous on the interval $[0, T]$ as $\mathbb{C}[0, T]$. On that space we define the following sup − metric

$$\rho_T(f, g) := \sup_{t \in [0,T]} \frac{|f(t) - g(t)|}{1+t^\alpha},$$

where $\alpha > \frac{1}{2}$; and the uniform metric

$$\rho_{T,U}(f, g) := \sup_{t \in [0,T]} |f(t) - g(t)|.$$

The rest of the paper consists of Section 2 and Section 3. In Section 2 we prove the main result. In Section 3 we formulate and prove auxiliary results.

2 Proof of the main result

Proof. We divide the proof into two steps.

Step 1. Let us prove that the set of limit points of the sequence w_n is contained in K a.s. To this end we show that the conditions of Lemma 2 are met for w_n a.s.

From [1, appendix 8, Theorem 4] and the inequality $\rho_T\left(w_{n_r(c)}, K_T\right) \leqslant \rho_{T,U}\left(w_{n_r(c)}, K_T\right)$ it follows that for any $c > 1$ and $T > 0$ the following equality holds

$$\mathbf{P}\left(\limsup_{r \to \infty} \rho_T\left(w_{n_r(c)}, K_T\right) = 0 \right) = 1, \tag{3}$$

where $n_r(c) = \lfloor c^r \rfloor$. That is, the condition (1) of Lemma 2 is satisfied a.s.

Now we show that for any $\varepsilon > 0$ there is $T_\varepsilon > 0$ such that for any $c > 1$ the following equality holds

$$\mathbf{P}\left(\limsup_{r \to \infty} \left(\sup_{t \geqslant T_\varepsilon} \frac{\max_{s \in [0,t]} \left| w_{n_r(c)}(s) \right|}{1+t^\alpha} \right) < \varepsilon \right) = 1. \tag{4}$$

For $T > 0$ we define a set of events as follows

$$A_r := \left\{ \sup_{t \geqslant T} \frac{\max_{s\in[0,t]} \left|w_{n_r(c)}(s)\right|}{1+t^\alpha} \geqslant \varepsilon \right\}, \quad r \in \mathbb{N}.$$

Using Lemma 4, we obtain the following bound

$$\sum_{r=1}^{\infty} \mathbf{P}(A_r) \leqslant C(T,\varepsilon) \sum_{r=1}^{\infty} 2\exp\left\{ \frac{-T^{2\alpha-1}\varphi^2(n_r(c))\varepsilon^2}{4} \right\}$$

$$= C(T,\varepsilon) \sum_{r=1}^{\infty} \frac{2}{(\ln(3 \vee n_r(c)))^{T^{2\alpha-1}\varepsilon^2/2}},$$

from which it follows that the series $\sum_{r=1}^{\infty} \mathbf{P}(A_r)$ converges if

$$T_\varepsilon > \left(\frac{2}{\varepsilon^2}\right)^{1/(2\alpha-1)}.$$

Therefore, from Borel–Cantelli Lemma it follows that the equality (4) holds, and consequently the condition (2) of Lemma 2 is satisfied a.s.

Thus, all the conditions of Lemma 2 are satisfied a.s., and we can conclude that the set of limit points of the sequence w_n is contained in K a.s.

Step 2. It remains to show that for any function $f \in K$ and almost all $\omega \in \Omega$ there is a sequence $n_k(\omega)$ such that $\lim_{k\to\infty} \rho(w_{n_k(\omega)}, f) = 0$.

We fix $\varepsilon > 0$. Let us choose $T_{1,\varepsilon}$ such that the following equality holds

$$\mathbf{P}\left(\limsup_{n\to\infty} \left(\sup_{t \geqslant T_\varepsilon} \frac{\max_{s\in[0,t]} |w_n(s)|}{1+t^\alpha} \right) < \varepsilon \right) = 1. \tag{5}$$

We recall that such choice is possible due to the equality (4) and the Lemma 1.

Let $f \in K$. For any $T > 0$ the following bound holds (see 2)

$$|f(T)| \leqslant \sqrt{T}.$$

Therefore we can choose $T_{2,\varepsilon} > 0$ such that for any $T \geqslant T_{2,\varepsilon}$ the following holds

$$\sup_{t \geqslant T} \frac{|f(T)|}{1+t^\alpha} < \varepsilon. \tag{6}$$

Since f is in K, for any $\varepsilon > 0$ there is $T_{3,\varepsilon} > 0$ such that

$$\int_{T_{3,\varepsilon}}^{\infty} \dot{f}^2(s)ds \leqslant \varepsilon^2. \tag{7}$$

It is easy to see that for $T_\varepsilon := \max(T_{1,\varepsilon}, T_{2,\varepsilon}, T_{3,\varepsilon})$ the inequalities (5), (6) and (7) hold simultaneously.

Using the Cauchy–Bunyakovsky–Schwarz inequality we get

$$\sup_{t\geqslant T_\varepsilon}\frac{|f(t)-f(T_\varepsilon)|}{1+t^\alpha}\leqslant \sup_{t\geqslant T_\varepsilon}\frac{\int_{T_\varepsilon}^t|\dot f(s)|ds}{1+t^\alpha}\leqslant \sup_{t\geqslant T_\varepsilon}\int_{T_\varepsilon}^t\frac{|\dot f(s)|}{1+s^\alpha}ds$$
$$\leqslant \int_{T_\varepsilon}^\infty\frac{|\dot f(s)|}{1+s^\alpha}ds\leqslant\left(\int_{T_\varepsilon}^\infty \dot f^2(s)ds\right)^{1/2}\cdot\left(\int_{T_\varepsilon}^\infty\frac{ds}{(1+s^\alpha)^2}\right)^{1/2}\leqslant \varepsilon C_\alpha. \quad (8)$$

From inequalities (6) and (8) it follows that

$$\rho(w_n,f)\leqslant \sup_{t\in[0,T_\varepsilon]}\frac{|w_n(t)-f(t)|}{1+t^\alpha}+\sup_{t\geqslant T_\varepsilon}\frac{|w_n(t)-f(t)|}{1+t^\alpha}\leqslant \sup_{t\in[0,T_\varepsilon]}\frac{|w_n(t)-f(t)|}{1+t^\alpha}$$
$$+\sup_{t\geqslant T_\varepsilon}\frac{|f(t)-f(T_\varepsilon)|}{1+t^\alpha}+\sup_{t\geqslant T_\varepsilon}\frac{|w_n(t)|}{1+t^\alpha}+\sup_{t\geqslant T_\varepsilon}\frac{|f(T_\varepsilon)|}{1+t^\alpha}$$
$$\leqslant \sup_{t\in[0,T_\varepsilon]}\frac{|w_n(t)-f(t)|}{1+t^\alpha}+\sup_{t\geqslant T_\varepsilon}\frac{|w_n(t)|}{1+t^\alpha}+(1+C_\alpha)\varepsilon. \quad (9)$$

Using the equality (5) we can conclude that for almost all $\omega\in\Omega$ there is $n(\varepsilon,\omega)$ such that for all $n\geqslant n(\varepsilon,\omega)$ the following equality holds

$$\sup_{t\geqslant T_\varepsilon}\frac{|w_n(t)|}{1+t^\alpha}<2\varepsilon. \quad (10)$$

From [1, appendix 8, Theorem 4] (the law of iterated logarithm for the Wiener process on the interval $[0,T_\varepsilon]$) it follows that for any function $f\in K_{T_\varepsilon}$ and almost all $\omega\in\Omega$ exists a sequence $n_k(\omega)$ such that $\lim\limits_{k\to\infty}\rho_{T_\varepsilon,U}(w_{n_k(\omega)},f)=0$. Therefore for almost all $\omega\in\Omega$

$$\lim_{k\to\infty}\sup_{t\in[0,T_\varepsilon]}\frac{|w_{n_k(\omega)}(t)-f(t)|}{1+t^\alpha}\leqslant \lim_{k\to\infty}\rho_{T_\varepsilon,U}(w_{n_k(\omega)},f)=0. \quad (11)$$

From (9), (10) and (11) it follows that for any function $f\in K$, $\varepsilon>0$ and almost all $\omega\in\Omega$ there is a subsequence $n_k(\omega)$ such that

$$\limsup_{k\to\infty}\rho(w_{n_k(\omega)},f)<(3+C_\alpha)\varepsilon.$$

The Theorem 1 is fully proven. □

3 Auxiliary results

For function $g\in\mathbb{C}$ we set

$$g_n(t):=\frac{g(nt)}{\sqrt{n}\varphi(n)},\quad n\in\mathbb{N}.$$

Lemma 1. *Let $g\in\mathbb{C}$ and let the following inequality hold for some $T>0$*

$$\limsup_{r\to\infty}\left(\sup_{t\geqslant T}\frac{\max\limits_{s\in[0,t]}|g_{2^r}(s)|}{1+t^\alpha}\right)<\varepsilon. \quad (12)$$

Then for the same $T>0$ the following inequality holds

$$\limsup_{n\to\infty}\left(\sup_{t\geqslant T}\frac{\max\limits_{s\in[0,t]}|g_n(s)|}{1+t^\alpha}\right)<2\varepsilon.$$

Proof. Since the inequality in condition (12) is strict, there is $r(\varepsilon)$ such that for all $r\geqslant r(\varepsilon)$ the following holds

$$\sup_{t\geqslant T}\frac{\max\limits_{s\in[0,t]}|g_{2^r}(s)|}{1+t^\alpha}<\varepsilon,\qquad \frac{\varphi(2^{r+1})}{\varphi(2^r)}<\sqrt{2}. \tag{13}$$

It is easy to see, that for any $n\geqslant 2^{r(\varepsilon)}$ there is $r(n)\geqslant r(\varepsilon)$ such that $2^{r(n)}\leqslant n<2^{r(n)+1}$. Therefore, using inequalities (13), for $n\geqslant 2^{r(\varepsilon)}$ we get

$$\begin{aligned}\sup_{t\geqslant T}\frac{\max\limits_{s\in[0,t]}|g_n(s)|}{1+t^\alpha}&=\sup_{t\geqslant T}\frac{\max\limits_{s\in[0,t]}|g(ns)|}{(1+t^\alpha)\sqrt{n}\varphi(n)}\leqslant\sup_{t\geqslant T}\frac{\max\limits_{s\in[0,t]}\left|g(2^{r(n)+1}s)\right|}{(1+t^\alpha)\sqrt{2^{r(n)}}\varphi(2^{r(n)})}\\&=\frac{\sqrt{2^{r(n)+1}}\varphi(2^{r(n)+1})}{\sqrt{2^{r(n)}}\varphi(2^{r(n)})}\cdot\sup_{t\geqslant T}\frac{\max\limits_{s\in[0,t]}\left|g(2^{r(n)+1}s)\right|}{(1+t^\alpha)\sqrt{2^{r(n)+1}}\varphi(2^{r(n)+1})}\\&=\frac{\sqrt{2}\varphi(2^{r(n)+1})}{\varphi(2^{r(n)})}\cdot\sup_{t\geqslant T}\frac{\max\limits_{s\in[0,t]}|g_{2^{r(n)+1}}(s)|}{1+t^\alpha}<2\varepsilon.\end{aligned}$$

□

Lemma 2. *Let $g\in\mathbb{C}$ and let it satisfy the following conditions:*

(1) *for any $c>1$ and $T>0$*

$$\limsup_{r\to\infty}\rho_T\left(g_{n_r(c)},K_T\right)=0,$$

where $n_r(c)=\lfloor c^r\rfloor$, $r\in\mathbb{N}$, $\rho_T\left(g_{n_r(c)},K_T\right):=\inf\limits_{f\in K_T}\rho_T\left(g_{n_r(c)},f\right)$;

(2) *for any $\varepsilon>0$ there is $T_\varepsilon>0$ such that for any $c>1$*

$$\limsup_{r\to\infty}\left(\sup_{t\geqslant T_\varepsilon}\frac{\max\limits_{s\in[0,t]}\left|g_{n_r(c)}(s)\right|}{1+t^\alpha}\right)<\varepsilon. \tag{14}$$

Then

$$\limsup_{n\to\infty}\rho\left(g_n,K\right)=0.$$

Proof. It is suffices to show that for any $\varepsilon>0$

$$\limsup_{n\to\infty}\rho\left(\frac{g(nt)}{\sqrt{n}\varphi(n)},K\right)<3\varepsilon. \tag{15}$$

Fix $\varepsilon>0$. Let us denote $C_{T_\varepsilon}:=\{f\in\mathbb{C}:f(t)\equiv f(T_\varepsilon),\ \text{for } t\geqslant T_\varepsilon\}$. It is obvious, that

$$\widetilde{K}_\varepsilon:=K_{T_\varepsilon}\cap C_{T_\varepsilon}\subset K, \tag{16}$$

and, due to the Cauchy–Bunyakovsky–Schwarz inequality, for any function $f \in K_{T_\varepsilon}$ the following equality holds

$$\sup_{t\in[0,T_\varepsilon]} |f(t)| \leqslant \int_0^{T_\varepsilon} |\dot f(s)|ds \leqslant \sqrt{T_\varepsilon}\left(\int_0^{T_\varepsilon} \dot f^2(s)ds\right)^{1/2} \leqslant \sqrt{T_\varepsilon}.$$

Therefore for any T_ε large enough the following inequality holds for $f \in \widetilde{K}_\varepsilon$

$$\sup_{t\geqslant T_\varepsilon} \frac{\max\limits_{s\in[0,t]} |f(s)|}{1+t^\alpha} = \sup_{t\geqslant T_\varepsilon} \frac{\max\limits_{s\in[0,T_\varepsilon]} |f(s)|}{1+t^\alpha} \leqslant \frac{\sqrt{T_\varepsilon}}{1+T_\varepsilon^\alpha} < \varepsilon. \tag{17}$$

It is easy to see, that if the inequality (14) holds for some $T_\varepsilon > 0$, then it also holds for any $T > T_\varepsilon$. For this reason, from now on, we will assume that T_ε is chosen such that both inequalities (14) and (17) hold.

Using (16) and (17) we get

$$\begin{aligned}
\limsup_{n\to\infty} \rho(g_n, K) &\leqslant \limsup_{n\to\infty} \rho(g_n, \widetilde{K}_\varepsilon)\\
&\leqslant \limsup_{n\to\infty} \inf_{f\in\widetilde{K}_\varepsilon} \left(\sup_{t\in[0,T_\varepsilon]} \frac{|g_n(t)-f(t)|}{1+t^\alpha} + \sup_{t\geqslant T_\varepsilon} \frac{|g_n(t)-f(t)|}{1+t^\alpha}\right)\\
&\leqslant \limsup_{n\to\infty} \inf_{f\in\widetilde{K}_\varepsilon} \left(\sup_{t\in[0,T_\varepsilon]} \frac{|g_n(t)-f(t)|}{1+t^\alpha} + \sup_{t\geqslant T_\varepsilon} \frac{\max\limits_{s\in[0,t]} |g_n(s)|}{1+t^\alpha} + \varepsilon\right)\\
&\leqslant \limsup_{n\to\infty} \rho_{T_\varepsilon}(g_n, K_{T_\varepsilon}) + \limsup_{n\to\infty} \left(\sup_{t\geqslant T_\varepsilon} \frac{\max\limits_{s\in[0,t]} |g_n(s)|}{1+t^\alpha}\right) + \varepsilon. \qquad (18)
\end{aligned}$$

It is easy to see that $\rho_{T_\varepsilon}(g_n, K_{T_\varepsilon}) \leqslant \rho_{T_\varepsilon,U}(g_n, K_{T_\varepsilon}) \leqslant (1+T_\varepsilon^\alpha)\rho_{T_\varepsilon}(g_n, K_{T_\varepsilon})$. We also note that the set K_{T_ε} is compact in the metric space $(\mathbb{C}[0,T_\varepsilon], \rho_{T_\varepsilon,U})$ (see, for example, [1, appendix 8, Lemma 2]). Therefore from condition (1) and [1, appendix 8, Lemma 6] it follows that

$$\limsup_{n\to\infty} \rho_{T_\varepsilon}(g_n, K_{T_\varepsilon}) = 0. \tag{19}$$

Applying condition (2) and Lemma 1, we get

$$\limsup_{n\to\infty} \left(\sup_{t\geqslant T_\varepsilon} \frac{\max\limits_{s\in[0,t]} |g_n(s)|}{1+t^\alpha}\right) < 2\varepsilon. \tag{20}$$

Inequality (15) follows from formulas (18)–(20). □

Lemma 3. *Let the function $f(t)$ be continuous on $[0,\infty)$, and let $\varepsilon > 0$. Then*

$$\limsup_{t\to\infty} \frac{\max\limits_{s\in[0,t]} |f(s)|}{1+t^\alpha} \geqslant \varepsilon \tag{21}$$

if and only if

$$\limsup_{t\to\infty}\frac{|f(t)|}{1+t^{\alpha}}\geqslant\varepsilon. \tag{22}$$

Proof. It is easy to see that (21) follows from (22).

Now we show that from the inequality

$$\limsup_{t\to\infty}\frac{|f(t)|}{1+t^{\alpha}}<\varepsilon, \tag{23}$$

it follows that

$$\limsup_{t\to\infty}\frac{\max\limits_{s\in[0,t]}|f(s)|}{1+t^{\alpha}}<\varepsilon. \tag{24}$$

Suppose that (23) holds, but (24) does not. Then there are sequences s_m and t_m such that $0\leqslant s_m\leqslant t_m$ for all $m\in\mathbb{N}$, $\lim\limits_{m\to\infty}t_m=\infty$ and

$$\lim_{m\to\infty}\frac{|f(s_m)|}{1+t_m^{\alpha}}\geqslant\varepsilon. \tag{25}$$

If $\liminf\limits_{m\to\infty}s_m<\infty$, then due to the continuity of the function f the inequality (25) cannot hold. Therefore $\lim\limits_{m\to\infty}s_m=\infty$. Then, due to the fact, that $s_m\leqslant t_m$, we have

$$\varepsilon\leqslant\lim_{m\to\infty}\frac{|f(s_m)|}{1+t_m^{\alpha}}\leqslant\lim_{m\to\infty}\frac{|f(s_m)|}{1+s_m^{\alpha}}\leqslant\limsup_{t\to\infty}\frac{|f(t)|}{1+t^{\alpha}}<\varepsilon.$$

The resulting contradiction completes the proof. □

Lemma 4. *For any $n\in\mathbb{N}$, $T>0$ and $\varepsilon>0$ the following bound holds*

$$\mathbf{P}\left(\sup_{t\geqslant T}\frac{\max\limits_{s\in[0,t]}|w_n(s)|}{1+t^{\alpha}}\geqslant\varepsilon\right)\leqslant 2\exp\left\{-\frac{T^{2\alpha-1}\varphi^2(n)\varepsilon^2}{4}\right\}C(T,\varepsilon),$$

where the constant $0<C(T,\varepsilon)<\infty$ depends only on T and ε.

Proof. Let $0<\lambda\in\mathbb{R}$. Applying Doob's martingale inequality (see., for example, [1, chapter 4, Corollary 5]) to the martingale $\exp\left\{\lambda\tilde{w}(t)-\frac{\lambda^2}{2}t\right\}$, and using the fact, that the random process $-\tilde{w}(t)$ is a Wiener process, for any $x>0$ and $y>0$ we get

$$\begin{aligned}\mathbf{P}\left(\max_{t\in[0,y]}|\tilde{w}(t)|\geqslant x\right)&\leqslant\mathbf{P}\left(\max_{t\in[0,y]}\tilde{w}(t)\geqslant x\right)+\mathbf{P}\left(\max_{t\in[0,y]}(-\tilde{w}(t))\geqslant x\right)\\ &\leqslant 2\mathbf{P}\left(\max_{t\in[0,y]}\exp\left\{\lambda\tilde{w}(t)-\frac{\lambda^2}{2}t\right\}\geqslant\exp\left\{\lambda x-\frac{\lambda^2}{2}y\right\}\right)\\ &\leqslant\frac{2\mathbf{E}\exp\left\{\lambda\tilde{w}(y)-\frac{\lambda^2}{2}y\right\}}{\exp\left\{\lambda x-\frac{\lambda^2}{2}y\right\}}=2\exp\left\{-\lambda x+\frac{\lambda^2}{2}y\right\}.\end{aligned}$$

By choosing $\lambda = \frac{x}{y}$, we will have

$$\mathbf{P}\left(\max_{t\in[0,y]}|\tilde{w}(t)| \geqslant x\right) \leqslant 2\exp\left\{-\frac{x^2}{2y}\right\}. \tag{26}$$

Using inequality (26), Lemma 3 and the Khinchin's law of iterated logarithm, we get

$$\mathbf{P}\left(\sup_{t\geqslant T}\frac{\max\limits_{s\in[0,t]}|w_n(s)|}{1+t^\alpha} \geqslant \varepsilon\right) = \mathbf{P}\left(\sup_{t\geqslant T}\frac{\max\limits_{s\in[0,t]}|\tilde{w}(s)|}{(1+t^\alpha)\varphi(n)} \geqslant \varepsilon\right)$$

$$\leqslant \mathbf{P}\left(\bigcup_{r=1}^{\infty}\left\{\sup_{t\in[Tr,T(r+1)]}\frac{\max\limits_{s\in[0,t]}|\tilde{w}(s)|}{(1+t^\alpha)} \geqslant \varphi(n)\varepsilon\right\}\right) + \mathbf{P}\left(\limsup_{t\to\infty}\frac{\max\limits_{s\in[0,t]}|\tilde{w}(s)|}{(1+t^\alpha)\varphi(n)} \geqslant \varepsilon\right)$$

$$= \mathbf{P}\left(\bigcup_{r=1}^{\infty}\left\{\sup_{t\in[Tr,T(r+1)]}\frac{\max\limits_{s\in[0,t]}|\tilde{w}(s)|}{(1+t^\alpha)} \geqslant \varphi(n)\varepsilon\right\}\right) + \mathbf{P}\left(\limsup_{t\to\infty}\frac{|\tilde{w}(t)|}{(1+t^\alpha)\varphi(n)} \geqslant \varepsilon\right)$$

$$= \mathbf{P}\left(\bigcup_{r=1}^{\infty}\left\{\sup_{t\in[Tr,T(r+1)]}\frac{\max\limits_{s\in[0,t]}|\tilde{w}(s)|}{(1+t^\alpha)} \geqslant \varphi(n)\varepsilon\right\}\right)$$

$$\leqslant \sum_{r=1}^{\infty}\mathbf{P}\left(\sup_{t\in[Tr,T(r+1)]}\frac{\max\limits_{s\in[0,t]}|\tilde{w}(s)|}{(1+t^\alpha)} \geqslant \varphi(n)\varepsilon\right) \leqslant \sum_{r=1}^{\infty}\mathbf{P}\left(\frac{\max\limits_{s\in[0,T(r+1)]}|\tilde{w}(s)|}{(1+(Tr)^\alpha)} \geqslant \varphi(n)\varepsilon\right)$$

$$\leqslant 2\sum_{r=1}^{\infty}\exp\left\{-\frac{(1+(Tr)^\alpha)^2\varphi^2(n)\varepsilon^2}{2T(r+1)}\right\} = 2\sum_{r=1}^{\infty}\exp\left\{-\frac{(1+2(Tr)^\alpha+(Tr)^{2\alpha})\varphi^2(n)\varepsilon^2}{2T(r+1)}\right\}$$

$$= 2\sum_{r=1}^{\infty}\exp\left\{-\frac{(1+2(Tr)^\alpha+(Tr)^{2\alpha}+T^{2\alpha}r-T^{2\alpha}r)\varphi^2(n)\varepsilon^2}{2T(r+1)}\right\}$$

$$= 2\sum_{r=1}^{\infty}\exp\left\{-\frac{(1+2(Tr)^\alpha+(Tr)^{2\alpha}-T^{2\alpha}r)\varphi^2(n)\varepsilon^2}{2T(r+1)}\right\}\exp\left\{\frac{-T^{2\alpha}r\varphi^2(n)\varepsilon^2}{2T(r+1)}\right\}$$

$$\leqslant 2\sum_{r=1}^{\infty}\exp\left\{-\frac{(1+2(Tr)^\alpha+(Tr)^{2\alpha}-T^{2\alpha}r)\varepsilon^2 2\ln\ln 3}{2T(r+1)}\right\}\exp\left\{\frac{-T^{2\alpha-1}\varphi^2(n)\varepsilon^2}{4}\right\}$$

$$= 2\exp\left\{\frac{-T^{2\alpha-1}\varphi^2(n)\varepsilon^2}{4}\right\}\sum_{r=1}^{\infty}\exp\left\{-\frac{(1+2(Tr)^\alpha+(Tr)^{2\alpha}-T^{2\alpha}r)\varepsilon^2 2\ln\ln 3}{2T(r+1)}\right\}$$

$$= 2\exp\left\{\frac{-T^{2\alpha-1}\varphi^2(n)\varepsilon^2}{4}\right\}C(T,\varepsilon).$$

Lemma 4 is proven. □

References

[1] A.V. Bulinskii, A.N. Shiryaev, *Teoriya sluchainykh protsessov (Theory of Stochastic Processes)*, Fizmatlit, Moscow, 2005 (in Russian).

[2] J.D. Deuschel, D.W. Stroock, *Large deviations*, Academic, Boston, 1989. Zbl 0705.60029

[3] N. Gantert, *An Inversion of Strassen's Law of the Iterated Logarithm for Small Time*, Ann. Probab., **21**:2 (1993), 1045–1049. Zbl 0776.60099

[4] D.S. Budkov, S.Ya. Makhno, *Functional iterated logarithm law for a wiener process*, Theory Stoch. Process., **13(29)**:3 (2007), 22–28. Zbl 1199.60202

[5] F.C. Klebaner, A.V. Logachov, A.A. Mogulskii, *Large deviations for processes on half–line*, Electron. Commun. Probab., **20** (2015), Paper No. 75. Zbl 1329.60055

Egor Vladimirovich Efremov
Sobolev Institute of Mathematics,
4 Acad. Koptyug avenue,
630090, Novosibirsk, Russia
Email address: efremov.eg.v@yandex.ru

Artem Vasilhevich Logachov
Sobolev Institute of Mathematics,
4 Acad. Koptyug avenue,
630090, Novosibirsk, Russia
Email address: omboldovskaya@mail.ru