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CANONICAL 3-FOLDS IN $\mathbb{P}^2 \times \mathbb{P}^2$ FORMAT AND NUMERICAL $K3$ TRANSITIONS

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Abstract: We provide a conjectural extension of the web of canonical 3-folds by proving the existence of some new deformation families of canonical 3-folds embedded in weighted projective space $\mathbb{P}^7(w_0, \dots, w_7)$. These families can be presented by using the equations of Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$. We provide a table of candidate numerical $K3$ transitions that can relate these families to the existing web of canonical 3-folds.

Keywords: canonical 3-folds, weighted projective space, $\mathbb{P}^2 \times \mathbb{P}^2$ format, K3 transitions.

1 Introduction

A *canonical 3-fold* is a normal complex projective algebraic variety with an ample canonical divisor class K_X, with at worst, $\mathbb{Q}$-factorial canonical singularities. Since, by the Minimal Model Program, any 3-fold of general type has a unique canonical model, it suffices for birational classification

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to classify canonical 3-folds. The canonical model $\mathrm{Proj}(R(X, K_X))$ of the canonical graded ring

$$R(X, K_X) = \bigoplus_{m \geq 0} H^0(X, mK_X)$$

can be embedded in a weighted projective space.

The graded ring constructions of canonical models of 3-folds go back to Reid and Iano-Fletcher [10, 7]. In particular, they listed 23 families of canonical hypersurfaces, 59 codimension two and 37 codimension three families of weighted complete intersections. These lists were proved to be complete by Chen–Chen–Chen [5, Table 3]. Beyond complete intersections, the idea of *Gorenstein formats*, i.e., describing the equations of canonical 3-folds by using the structure of low codimension Gorenstein ideals, has been used to construct new examples of non complete intersection canonical 3-folds. In particular, Brown–Kasprzyk–Zhou produced 18 new families in Pfaffian $\mathrm{Gr}(2, 5)$ format, and 84 families as hypersurface in $\mathrm{Gr}(2, 5)$ format in [3]. In [6], Coughlan introduced $K3$ *transitions* as a geometric tool to navigate between deformation families of canonical 3-folds: one degenerates to a cone over a (polarized) $K3$ surface, resolves, and then takes the canonical model to land in a new family. Using this, the known geography has been shaped into a *web of canonical 3-folds* and 137 families in codimension at most 3 are listed, together with explicit $K3$ transitions between them.

In this note, we present a construction of nineteen deformation families of wellformed and quasismooth canonical 3-folds embedded in weighted projective space $\mathbb{P}^7(\mathbf{w})$ that can be described in terms of 2×2 minors of a 3×3 matrix, i.e. they are given by the image of $\mathbb{P}^2 \times \mathbb{P}^2$ under the Segre embedding, cf.[8]. To link the new families to the existing web of canonical 3-folds, in the sense of Coughlan, we list *candidate numerical* $K3$ *transitions* for each of our families of 3-folds, i.e for each canonical 3-fold X in $\mathbb{P}^2 \times \mathbb{P}^2$ format we identify the candidate $K3$ surfaces S and canonical 3-fold Y from the graded ring database [1] such that their baskets and volumes satisfy the conditions described in Theorem 4.2. The proof of Theorem 4.2 has not been published, but it is instrumental in proving the existence of $K3$ transitions in [6].

Theorem 1.1. *Let $X \subset \mathbb{P}(w_0, \dots, w_7)$ be a canonical 3-fold whose image under the canonical embedding can be described in terms of the equations of Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$. Then there exist at least 19 such families of canonical 3-folds with at worst terminal isolated orbifold points whose general member is wellformed and quasismooth, listed in Table 1. Among these 19 cases, 3 are smooth canonical 3-folds.*

Table 1: Canonical 3-folds in $\mathbb{P}^2 \times \mathbb{P}^2$ format

S.No	Variety	Basket B	K^3	χ	$K \cdot c_2$	Weight matrix
1	$X_{3^6,4^3} \subset \mathbb{P}(1^7,2)$		22	-6	144	$\begin{matrix}1&1&1\\2&2&2\\2&2&2\end{matrix}$
2	$X_{2,3^4,4^4} \subset \mathbb{P}(1^7,2)$		20	-6	144	$\begin{matrix}1&1&2\\1&1&2\\2&2&3\end{matrix}$
3	$X_{4^9} \subset \mathbb{P}(1^5,2^3)$		12	-4	96	$\begin{matrix}2&2&2\\2&2&2\\2&2&2\end{matrix}$
4	$X_{3^2,4^5,5^2} \subset \mathbb{P}(1^5,2^3)$	$2 \times \frac{1}{2}(1,1,1)$	11	-4	99	$\begin{matrix}1&1&2\\2&2&3\\2&2&3\end{matrix}$
5	$X_{3,4^3,5^3,6^2} \subset \mathbb{P}(1^4,2^3,3)$	$3 \times \frac{1}{2}(1,1,1)$	$\frac{13}{2}$	-3	$\frac{153}{2}$	$\begin{matrix}1&1&2\\2&2&3\\3&3&4\end{matrix}$
6	$X_{4^4,5^4,6} \subset \mathbb{P}(1^4,2^3,3)$	$2 \times \frac{1}{2}(1,1,1)$	7	-3	75	$\begin{matrix}1&2&2\\2&3&3\\2&3&3\end{matrix}$
7	$X_{4,5^4,6^4} \subset \mathbb{P}(1^3,2^3,3^2)$	$4 \times \frac{1}{2}(1,1,1)$	4	-2	54	$\begin{matrix}2&2&3\\2&2&3\\3&3&4\end{matrix}$
8	$X_{4^2,5^3,6^3,7} \subset \mathbb{P}(1^3,2^3,3^2)$	$3 \times \frac{1}{2}(1,1,1),\ \frac{1}{3}(1,2,2)$	$\frac{23}{6}$	-2	$\frac{331}{6}$	$\begin{matrix}1&2&2\\2&3&3\\3&4&4\end{matrix}$
9	$X_{5^2,6^5,7^2} \subset \mathbb{P}(1^2,2^3,3^3)$	$5 \times \frac{1}{2}(1,1,1),\ 2 \times \frac{1}{3}(1,2,2)$	$\frac{13}{6}$	-1	$\frac{221}{6}$	$\begin{matrix}2&2&3\\3&3&4\\3&3&4\end{matrix}$
10	$X_{4,5^2,6^3,7^2,8} \subset \mathbb{P}(1^3,2^2,3^2,4)$	$2 \times \frac{1}{2}(1,1,1)$	3	-2	51	$\begin{matrix}1&2&3\\2&3&4\\3&4&5\end{matrix}$

Continued on next page

S.No	Variety	Basket B	K^3	χ	$K\cdot c_2$	Weight matrix
11	$X_{4,5^2,6^3,7^2,8} \subset \mathbb{P}(1^2,2^3,3^3)$	$4\times\frac{1}{2}(1,1,1),\ 3\times\frac{1}{3}(1,2,2)$	2	-1	38	1 2 3 2 3 4 3 4 5
12	$X_{6^4,7^4,8} \subset \mathbb{P}(1^2,2^2,3^3,4)$	$2\times\frac{1}{2}(1,1,1),\ 2\times\frac{1}{3}(1,2,2)$	$\frac{5}{3}$	-1	$\frac{97}{3}$	2 3 3 3 4 4 3 4 4
13	$X_{5,6^2,7^2,8^2,9,10} \subset \mathbb{P}(1^2,2^2,3^2,4,5)$	$4\times\frac{1}{2}(1,1,1),\ \frac{1}{3}(1,2,2)$	$\frac{4}{3}$	-1	$\frac{98}{3}$	1 2 3 3 4 5 4 5 6
14	$X_{6,7^2,8^3,9^2,10} \subset \mathbb{P}(1^2,2,3^2,4^2,5)$	$2\times\frac{1}{2}(1,1,1)$	1	-1	27	2 3 4 3 4 5 4 5 6
15	$X_{6,7^2,8^3,9^2,10} \subset \mathbb{P}(1,2^2,3^3,4,5)$	$4\times\frac{1}{2}(1,1,1),\ 5\times\frac{1}{3}(1,2,2)$	$\frac{2}{3}$	0	$\frac{58}{3}$	2 3 4 3 4 5 4 5 6
16	$X_{8^2,9^3,10^3,11} \subset \mathbb{P}(1,2,3^2,4^2,5^2)$	$2\times\frac{1}{2}(1,1,1),\ \frac{1}{3}(1,2,2)$ $\frac{1}{4}(1,3,3),\ \frac{1}{5}(2,3,4)$	$\frac{23}{60}$	0	$\frac{853}{60}$	3 4 4 4 5 5 5 6 6
17	$X_{8,9^2,10^3,11^2,12} \subset \mathbb{P}(1,2,3^2,4,5^2,6)$	$2\times\frac{1}{2}(1,1,1),\ 4\times\frac{1}{3}(1,2,2)$	$\frac{1}{3}$	0	$\frac{41}{3}$	3 4 5 4 5 6 5 6 7
18	$X_{8,9^2,10^3,11^2,12} \subset \mathbb{P}(2,3^3,4^2,5^2)$	$2\times\frac{1}{2}(1,1,1),\ 8\times\frac{1}{3}(1,2,2)$ $2\times\frac{1}{4}(1,3,3)$	$\frac{1}{6}$	1	$\frac{47}{6}$	3 4 5 4 5 6 5 6 7
19	$X_{10,11^2,12,13^2,14} \subset \mathbb{P}(2,3^2,4,5^2,6,7)$	$3\times\frac{1}{2}(1,1,1),\ 6\times\frac{1}{3}(1,2,2)$ $2\times\frac{1}{5}(2,3,4)$	$\frac{1}{10}$	1	$\frac{61}{10}$	4 5 6 5 6 7 6 7 8

The proof requires the use of an algorithmic approach developed in [9, 3] to search for possible examples and then proving the existence of these canonical 3-folds. We outline the details in section 3.

Table notations. In this part, we explain the notations used in the Table 1. We list canonical 3-folds in their canonical embedding $X_{\mathbf{d}} \subset \mathbb{P}(\mathbf{w})$ in column "variety". Here $\mathbb{P}(\mathbf{w}) = \mathbb{P}(w_0, \ldots, w_7)$ is a weighted projective space and $\mathbf{d}$ are the degrees of the defining equations, written in compact form: For example, $X_{3^6,4^3} \subset \mathbb{P}(1^7, 2)$ means six cubics and three quartics define this 3-fold inside $\mathbb{P}^7(w_0, \ldots, w_7)$ with 7 variables of degree one and 1 variable of degree two. The column Basket contains a multiset of terminal cyclic quotient singularities

$$\mathcal{B} = \left\{ m_i \times \tfrac{1}{r_i}(a_i, b_i, c_i) \right\},$$

listed with multiplicities m_i. The holomorphic Euler characteristic $\chi = \chi(\mathcal{O}_X)$ gives the

$$\chi(\mathcal{O}_X) = \sum_{i=0}^{3} (-1)^i h^i(\mathcal{O}_X).$$

In column K^3, we list the self-intersection number (canonical volume) K_X^3 of X and the intersection number $K_X \cdot c_2(X)$ appears in the next column. The last column records the 3×3 array of variable weights used by the $\mathbb{P}^2 \times \mathbb{P}^2$ (Segre) Gorenstein format; it lists the degrees of the variables $(x_{ij} : 1 \leq i, j \leq 3)$ that appear in the equations of $\mathbb{P}^2 \times \mathbb{P}^2$.

2 Preliminaries

In this section, we provide some definitions and notations that are later used in the proofs.

A weighted projective variety $X \subset \mathbb{P}^N(\mathbf{w})$ of codimension c is *wellformed* if $\mathbb{P}^N(\mathbf{w})$ is wellformed, i.e. no $N-1$ weights have a non trivial common factor, and

$$\dim X - \dim \big(X \cap \operatorname{Sing}(\mathbb{P}^N(\mathbf{w})\big) \geq 2.$$

It is called *quasismooth* if its affine cone $\widetilde{X} \subset \mathbb{A}^{N+1}$ is smooth outside the origin. A *format* is a structured presentation of equations for a family of varieties. For example, the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ is defined by the 2×2 minors of a 3×3 matrix.

Definition 2.1 ([3])**.** A codimension c *Gorenstein format* $\mathcal{F}$ is a triple $(\widetilde{V}, \mathcal{R}, \mu)$ where $\widetilde{V} \subset \mathbb{A}^n$ is an affine Gorenstein subvariety of codimension c, $\mathcal{R}$ is a minimal graded free resolution of $\mathcal{O}_{\widetilde{V}}$ over $\mathcal{O}_{\mathbb{A}^n}$, and $\mu : \mathbb{C}^\times \circlearrowright \widetilde{V}$ is a grading by strictly positive weights. We assume μ preserves $\widetilde{V}$ and $\mathcal{R}$ is $\mathbb{C}^\times$-equivariant. (Definition 2.2 below fits this framework.)

Definition 2.2 ([2])**.** Let $\Sigma \subset \mathbb{P}^8(x_{ij})$ be the Segre image of $\mathbb{P}^2 \times \mathbb{P}^2$ for $1 \leq i, j \leq 3$. Let $t \in \mathbb{C}^\times$ act on the punctured affine cone $\widetilde{\Sigma} \setminus \{\underline{0}\}$ by

$$t \cdot x_{ij} = t^{e_i + f_j} x_{ij} \quad (1 \leq i, j \leq 3),$$

for non-negative integer vectors $e = (e_1, e_2, e_3)$ and $f = (f_1, f_2, f_3)$ with

$$e_i + f_j > 0, \qquad e_1 \le e_2 \le e_3, \qquad f_1 \le f_2 \le f_3.$$

Then the quotient is called a *weighted* $\mathbb{P}^2 \times \mathbb{P}^2$ variety, which we denote by $\mathrm{w}\mathcal{P}$. If $w_{ij} = e_i + f_j$, then we have

$$\mathrm{w}\mathcal{P} \hookrightarrow \mathbb{P}^8\big(w_{ij} : 1 \le i, j \le 3\big).$$

Its image is cut out by the 2×2 minors of the 3×3 matrix of variables, and we record the degrees via the *weight matrix*

$$\begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix}. \tag{2.1}$$

If $\mathrm{w}\mathcal{P}$ is wellformed, then its canonical divisor is

$$K_{\mathrm{w}\mathcal{P}} = \Big(-\sum_{i=1}^{3} w_{ii}\Big)\, D, \tag{2.2}$$

where $D = \mathcal{O}_{\mathrm{w}\mathcal{P}}(1)$ is the hyperplane class.

Theorem 2.3 (Bertini)**.** *Let $L = |\mathcal{O}_{\mathbb{P}(w_0,\dots,w_n)}(d)|$ be a linear system and let $X \in L$ be a general member. Then X is quasismooth away from the reduced base locus* $\mathrm{Bs}(L)_{\mathrm{red}}$.

3 Proof of Theorem 1.1

The proof of the theorem consists of a mix of algorithmic, computational, and theoretical ideas. We first find the list of candidate 3-folds in $\mathbb{P}^2 \times \mathbb{P}^2$ format that may contain at worst isolated terminal quotient singularities by using their Hilbert series data. Then, for each candidate, we prove or disprove the existence of a given variety in $\mathbb{P}^2 \times \mathbb{P}^2$ format.

3.1. Steps of the proof. In this section, we describe various steps involved in the proof of Theorem 1.1.

3.1.1. Finding isolated candidate orbifolds. We enumerate candidate canonical 3-folds with isolated terminal singularities using the algorithmic approach of [9], which applies the orbifold Riemann–Roch formula of Buckley–Reid–Zhou [4]. The formula states that if X has a basket $\mathcal{B} = \{k_i \times Q_i : k_i \in \mathbb{Z}_{>0}\}$ of isolated points, then

$$P_X(t) = P_{\mathrm{smooth}}(t) + \sum_{Q_i \in \mathcal{B}} k_i\, P_{Q_i}(t), \tag{3.1}$$

where P_{smooth} is the smooth contribution and P_{Q_i} are the local point contributions. In a given ambient weighted variety $\mathrm{w}\mathcal{F} \hookrightarrow \mathbb{P}^N(\mathbf{w})$ of dimension

$d \gg 0$ and fixed codimension c, the algorithm generates a complete list of orbifolds of dimension $n \leq d$, orbifold canonical class $K_X = \mathcal{O}(k)$, and specified type (terminal or canonical) of isolated orbifold points. The condition $d \gg 0$ means, for our purposes, at least $d = 4$ if we aim to construct canonical 3-folds.

First, compute the Hilbert series and the canonical divisor of the ambient weighted projective variety $\mathrm{w}\mathcal{F}$. Next, we enumerate embeddings $X \hookrightarrow \mathbb{P}^s(\mathbf{w})$, with $K_X = \mathcal{O}(k)$, by applying adjunction and finding appropriate combinations of weights for $\mathbb{P}^s(\mathbf{w})$. For each embedding, compute $P_X(t)$ and extract its smooth part $P_{\mathrm{smooth}}(t)$. From the weights of $\mathbb{P}^s(\mathbf{w})$, list all possible subsets of the isolated orbifold points that could lie on X. Then, for every subset $\mathcal{B}$, test whether

$$P_X(t) - P_{\mathrm{smooth}}(t) = \sum_{Q_i \in \mathcal{B}} k_i \, P_{Q_i}(t)$$

for suitable integers k_i. Finally, accept X as a candidate n-fold with basket $\mathcal{B}$ precisely when all resulting coefficients are non-negative integers, and repeat this over all embeddings.

Candidate canonical 3-folds in $\mathbb{P}^2 \times \mathbb{P}^2$ format. For each choice of parameters $e = (e_1, e_2, e_3)$ and $f = (f_1, f_2, f_3)$, we obtain a weighted $\mathbb{P}^2 \times \mathbb{P}^2$ variety

$$\mathrm{w}\mathcal{P} \hookrightarrow \mathbb{P}\big(w_{ij} = e_i + f_j \mid 1 \leq i, j \leq 3\big).$$

Since there is no bound on the search parameters, the search is, in principle, infinite. We order the search by increasing total weight, where $W := \sum_{i,j} w_{ij}$ and continue until limited by computer memory. In practice, the search halts at $W = 96$, with the final candidate already appearing at $W = 54$. It is therefore reasonable to conjecture that the list in Table 1 is complete. Then, for each candidate, we perform the following steps to check for well-formedness and quasismoothness. In total, we obtain 21 candidate canonical 3-folds with terminal quotient singularities, of which 19 are well-formed and quasismooth, while two fail quasismoothness.

3.1.2. Existence and well-formedness. For each of the 21 candidates, we verify that the baskets of candidates predicted in 3.1.1 matches the actual singularities by intersecting X with the toric orbifold strata of $\mathbb{P}^n(w)$, usually using the computer algebra system Magma. Since we only allow isolated orbifold points, every such intersection must be zero-dimensional; this also enforces well-formedness. We discard a candidate if any orbifold point is non-terminal or if X meets a higher-dimensional orbifold locus.

3.1.3. Quasismoothness. The step of proving quasismoothness is usually the hardest among these steps. Taking complete intersections of degree d inside $\mathrm{w}\mathcal{P}$ induces a base locus of $|\mathcal{O}(d)|$ that can be large due to weighted degrees. By Bertini's theorem, a general member is quasismooth away from the reduced base locus; if that locus is zero-dimensional, theoretical arguments suffice. In the more common higher-dimensional case, we use Magma to write defining equations over the rational numbers and apply the Jacobian criterion to prove quasismoothness.

3.2. Sample Computations. In this section, we record some of the computations used to establish the existence of canonical 3-folds. As illustrations, we present two cases (Examples 4 and 10) from the table. For Examples 1–9, the base locus coincides with the orbifold locus, thus quasismoothness can be checked directly. In the remaining cases, the base locus is more involved, we rely on computer algebra calculations to verify quasismoothness.

Example 4: $X_{3^2,4^5,5^2} \subset \mathbb{P}(1^5, 2^3)$**.** Let $\mathrm{w}\mathcal{P} \subset \mathbb{P}^8$ be the weighted Segre 4-fold defined by the 2×2 minors of the 3×3 matrix $X = (x_{ij})$ with weights

$$W = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix},$$

so the ambient has graded coordinates of weights $(1^2, 2^5, 3^2)$ and

$$K_{\mathrm{w}\mathcal{P}} = \mathcal{O}_{\mathrm{w}\mathcal{P}}\big(-\sum_{i=1}^{3} w_{ii}\big) = \mathcal{O}_{\mathrm{w}\mathcal{P}}(-6).$$

We adjoin three new weight 1 variables that do not appear in the weight matrix, to form the *triple projective cone*

$$V := \mathcal{C}^3\mathrm{w}\mathcal{P} \hookrightarrow \mathbb{P}(1^5, 2^5, 3^2), \qquad K_V = \mathcal{O}_V(-9).$$

The intersection of V with two general cubics and two general quadrics, by the adjunction formula, gives a canonical 3-fold $X \subset \mathbb{P}(1^5, 2^3)$.

Orbifold locus. The weight 2 locus intersects with X in 2 isolated points, which we can show to be of type $\frac{1}{2}(1,1,1)$.

Base locus. On $\mathbb{P}(1^5, 2^5, 3^2)$ the linear system $|\mathcal{O}(3)|$ has base locus $\mathbb{P}(2^5)$ (the coordinate subspace on which no degree 3 monomial can be formed). Hence

$$V_1 := \{V \cap (f_3) \cap (g_3)\} \subset \mathbb{P}(1^5, 2^5)$$

is quasismooth away from $\mathbb{P}(2^5)$ by a Bertini's theorem. Intersecting further with two general quadrics, the 3-fold

$$X := \{V_1 \cap (f_2) \cap (g_2)\} \subset \mathbb{P}(1^5, 2^3)$$

meets $\mathbb{P}(2^3)$ in exactly two points. A local Jacobian (implicit function) check at each point shows that three suitably chosen tangent variables of odd residue mod 2 give full rank, so both points are terminal cyclic quotients of type $\frac{1}{2}(1,1,1)$. Away from these two points the Jacobian has full rank by a Bertini's theorem, whence X is quasismooth.

Example 10: $X_{4,5^2,6^3,7^2,8} \subset \mathbb{P}(1^3, 2^2, 3^2, 4)$**.** Let $\mathrm{w}\mathcal{P} \subset \mathbb{P}^8$ be the weighted Segre 4–fold defined by the 2×2 minors of a 3×3 matrix $X = (x_{ij})$ with weights

$$W = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix},$$

so the ambient has graded coordinates of weights $(1, 2^2, 3^3, 4^2, 5)$ and

$$K_{\mathrm{w}\mathcal{P}} = \mathcal{O}_{\mathrm{w}\mathcal{P}}\left(-\sum_{i=1}^{3} w_{ii}\right) = \mathcal{O}_{\mathrm{w}\mathcal{P}}(-9).$$

We adjoin two new weight 1 variables that do not appear in the equations of $\mathrm{w}\mathcal{P}$, to form the *double projective cone* over $\mathrm{w}\mathcal{P}$, given by

$$V := \mathcal{C}^2\mathrm{w}\mathcal{P} \hookrightarrow \mathbb{P}(1^3, 2^2, 3^3, 4^2, 5), \qquad K_V = \mathcal{O}_V(-11).$$

Intersecting V with one general quintic, one general quartic, and one general cubic, by adjunction, gives a canonical 3-fold

$$X := V \cap (5) \cap (3) \cap (2)^2 \subset \mathbb{P}(1^3, 2^2, 3^2, 4),$$

as $-11 + 5 + 4 + 3 = 1$.

Orbifold locus. We analyze the orbifold strata of the ambient $\mathbb{P}(\mathbf{w})$ that could meet X. The weight 4 stratum is the coordinate point, however this point does not lie on X. Likewise, the stratum $\mathbb{P}(3,3)$ is disjoint from X: restricting the equations to $\mathbb{P}(3,3)$ eliminates both weight 3 variables, so the intersection is empty. The only relevant orbifold stratum is $\mathbb{P}(2,2,4)$, which parametrizes the locus where all variables of weight $\neq 2, 4$ vanish. Its intersection with the defining equations of X cuts this stratum in exactly two reduced points. A local Jacobian computation shows that in suitable affine coordinates each of these is analytically isomorphic to $\frac{1}{2}(1,1,1)$, so they are terminal cyclic quotient singularities. There are no other orbifold points on

X. Thus X is well-formed canonical 3-fold with the basket of singularities

$$\mathcal{B} = \{\, 2 \times \tfrac{1}{2}(1,1,1)\, \}.$$

Base locus. We analyze the base locus of the intersections with forms of various degrees, stepwise. We first take intersection of $V = \mathcal{C}^2\mathrm{w}\mathcal{P}$ with a general quartic:

$$V_1 := \{V \cap (f_4)\} \;\subset\; \mathbb{P}(1^3, 2^2, 3^3, 4, 5).$$

Then by a Bertini's theorem V_1 is quasismooth away from the base locus of $|\mathcal{O}(4)|$, namely $V_1 \cap \mathbb{P}(3^3, 5)$, since degrees 4 monomials cannot contain pure powers of weight 3 and weight 5 variables. Next, we intersect V_1 with a general quintic:

$$V_2 := \{V_1 \cap (f_5)\} \subset\; \mathbb{P}(1^3, 2^2, 3^3, 4).$$

On V_2 the base locus of $|\mathcal{O}(5)|$ is $V_2 \cap \mathbb{P}(2^2, 3^3, 4)$, because degree 5 monomials cannot be formed with pure powers of weights $2, 3, 4$. The intersection of this base locus with V_2 is positive dimensional and we can not determine the quasismoothness by analyzing the orbifold points on the locus. We finally set

$$X := \{V_2 \cap (f_3)\} \;\subset\; \mathbb{P}(1^3, 2^2, 3^2, 4).$$

Here the residual base locus of $|\mathcal{O}(3)|$ meets X along $\mathbb{P}(2^2, 4)$, which for a general cubic consists of exactly two reduced points. However, the base locus of V_2 is too large and so quasismoothness cannot be established purely by the implicit function theorem. We need to use computer algebra to show the quasismoothness and use the Jacobian criterion over $\mathbb{Q}$ in MAGMA for random choices of coefficients; the computation certifies quasismoothness of a general member. We can use the following code to show the quasismoothness for this example.[1]

```
rpoly := func< P,d | d ge 0 select
&+[ Random([1..10])*m : m in MonomialsOfWeightedDegree(CoordinateRing(P),d)]
else CoordinateRing(P)!0 >;
P<x0,x1,x11,x12,x13,x21,x22,x23>:=ProjectiveSpace(Rationals(),[1,1,1,2,3,2,3,4]);
f3:=rpoly(P,3);f4:=rpoly(P,4);f5:=rpoly(P,5);
M := Matrix(3,3,[
x11,x12,x13,
x21, x22,x23,
f3, f4,f5
]);
X := Scheme(P,Minors(M,2));
```

[1]We provide codes for examples 10–21 on https://github.com/QureshiMI/P2-x-P2-C3F.

```
Dim:=Dimension(JacobianSubrankScheme(X));
print "Dimension of Jacobian subrank scheme=",Dim;
```

3.3. Non working candidates. It is an essential part of the calculation to list the non working canonical 3-folds in the $\mathbb{P}^2 \times \mathbb{P}^2$ model and write the reason for failure. The first one fails as it contains the non-quasimooth $\frac{1}{5}$ point and the second one contains a non-terminal point of type $\frac{1}{4}(1,1,1)$.

Table 2: Non-working canonical 3-fold candidates in $\mathbb{P}^2 \times \mathbb{P}^2$ format

S.No	Variety	Basket B	K^3	χ	$K \cdot c_2$	Weight matrix
1	$X_{7,8^2,9^2,10^2,11,12} \subset \mathbb{P}(1,2,3^2,4^2,5^2)$	$3 \times \frac{1}{2}(1,1,1), 2 \times \frac{1}{3}(1,2,2), \frac{1}{5}(1,4,4)$	$\frac{11}{30}$	0	$\frac{439}{30}$	$\begin{matrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{matrix}$
2	$X_{6,8^2,9^2,10,11^2,12} \subset \mathbb{P}(1,2,3^2,4^2,5^2)$	$\frac{1}{2}(1,1,1), \frac{1}{4}(1,3,3), 2 \times \frac{1}{5}(2,3,4)$	$\frac{7}{20}$	0	$\frac{297}{20}$	$\begin{matrix} 2 & 3 & 5 \\ 3 & 4 & 6 \\ 5 & 6 & 8 \end{matrix}$

4 Candidate numerical K3 Transitions

In this section, we recall the definition of $K3$ transition and provide a table that lists candidate $K3$ surfaces and canonical 3-folds, from graded ring database [1], that satisfy the properties listed in Theorem 4.2 for the existence of $K3$ transition. First we recall the notion of $K3$ transition.

Definition 4.1. [6] Let X be a quasismooth canonical 3-fold that degenerates in a flat family to a central fibre X_0 with a *simple elliptic* singularity P (the vertex of the affine cone over a K3 surface).

(i) Take the weighted blow-up $\sigma\colon Z \to X_0$ so that the exceptional divisor is a quasismooth K3 surface $S \subset \mathbb{P}(\alpha)$.

(ii) Form the canonical model

$$Y_0 = \operatorname{Proj}\Big(\bigoplus_{n\geq 0} H^0\big(Z, nK_Z\big)\Big);$$

the variety Y_0 has only ordinary double points along the image of S.

(iii) If Y_0 admits a (partial) smoothing to another quasismooth canonical 3-fold Y, the birational process

$$\begin{array}{ccccccc} & & & Z & & & \\ & & \sigma\swarrow & & \searrow\pi & & \\ X & \rightsquigarrow & X_0 & & Y_0 & \leftsquigarrow & Y \end{array}$$

is called a *K3 transition through the surface* S, written $X \rightsquigarrow Y$.

Theorem 4.2. [6, Thm. 2.1] *Let X and Y be quasismooth canonical 3-folds related by a K3 transition through a quasismooth K3 surface S.*

Then the following are satisfied

$$K_X^3 \;=\; K_Y^3 + A^2, \qquad p_g(X) \;=\; p_g(Y) + 1, \qquad \mathcal{B}(Y) \;=\; \mathcal{B}(X) \cup \mathcal{B}(S),$$

where

- K_X^3 *(resp. K_Y^3) is the self?intersection number of the canonical divisor on X (resp. Y) and $A = \mathcal{O}_S(1)$,*
- $p_g(\bullet) = h^0(\bullet, K_\bullet)$ *is the geometric genus,*
- $\mathcal{B}(\bullet)$ *denotes the basket of quotient singularities on the variety $\bullet$.*

Here X is the original canonical 3-fold before transition, S the exceptional $K3$ surface arising in the weighted blow-up of the singular fiber X_0, and Y the resulting canonical 3-fold after contracting and smoothing S. We say that the *candidate numerical* $K3$ *transition* exists between X and Y, if there exist a candidate $K3$-surface S satisfying all the properties listed in Theorem 4.2.

The Table 3 lists the graded ring database ID's for the candidate $K3$ surfaces and canonical 3-folds. The $K3$ surfaces listed under these ID's on the GRDB shall be treated as candidates since the existence of all of these are not proven. However, the canonical 3-folds listed below, do indeed exist as wellformed and quasismooth 3-folds. The DNE in the Table 3 refers to non-existing candidate for the given $\mathbb{P}^2 \times \mathbb{P}^2$ canonical 3-fold in the GRDB [1] and NA means that the $K3$ transition can not exist as the geometric genus $p_g = 0$ in these cases. Establishing the existence of these $K3$ transitions will be studied elsewhere.

Table 3: Candidate numerical $K3$ transitions

S.No	Variety X	GRDB K3 ID	GRDB C3F ID
1	$X_{3^6,4^3} \subset \mathbb{P}(1^7,2)$	24073	123

Continued on next page

S.No	Variety X	GRDB K3 ID	GRDB C3F ID
2	$X_{2,3^4,4^4} \subset \mathbb{P}(1^7, 2)$	24052	123
3	$X_{4^9} \subset \mathbb{P}(1^5, 2^3)$	24044	85
4	$X_{3^2,4^5,5^2} \subset \mathbb{P}(1^5, 2^3)$	24017	126
5	$X_{3,4^3,5^3,6^2} \subset \mathbb{P}(1^4, 2^3, 3)$	17408	128
6	$X_{4^4,5^4,6} \subset \mathbb{P}(1^4, 2^3, 3)$	17506	128
7	$X_{4,5^4,6^4} \subset \mathbb{P}(1^3, 2^3, 3^2)$	11157	131
8	$X_{4^2,5^3,6^3,7} \subset \mathbb{P}(1^3, 2^3, 3^2)$	11156	131
9	$X_{5^2,6^5,7^2} \subset \mathbb{P}(1^2, 2^3, 3^3)$	5939	95
10	$X_{4,5^2,6^3,7^2,8} \subset \mathbb{P}(1^3, 2^2, 3^2, 4)$	10879	133
11	$X_{4,5^2,6^3,7^2,8} \subset \mathbb{P}(1^2, 2^3, 3^3)$	5921	95
12	$X_{6^4,7^4,8} \subset \mathbb{P}(1^2, 2^2, 3^3, 4)$	5194	99
13	$X_{5,6^2,7^2,8^2,9,10} \subset \mathbb{P}(1^2, 2^2, 3^2, 4, 5)$	5094	135
14	$X_{6,7^2,8^3,9^2,10} \subset \mathbb{P}(1^2, 2, 3^2, 4^2, 5)$	4677	136
15	$X_{6,7^2,8^3,9^2,10} \subset \mathbb{P}(1, 2^2, 3^3, 4, 5)$	DNE	DNE
16	$X_{8^2,9^3,10^3,11} \subset \mathbb{P}(1, 2, 3^2, 4^2, 5^2)$	844	108

Continued on next page

S.No	Variety X	GRDB K3 ID	GRDB C3F ID
17	$X_{8,9^2,10^3,11^2,12} \subset \mathbb{P}(1,2,3^2,4,5^2,6)$	DNE	DNE
18	$X_{8,9^2,10^3,11^2,12} \subset \mathbb{P}(2,3^3,4^2,5^2)$	NA	NA
19	$X_{10,11^2,12,13^2,14} \subset \mathbb{P}(2,3^2,4,5^2,6,7)$	NA	NA

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