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ON ACCURACY OF APPROXIMATIONS FOR WIENER PROBABILITIES TO STAY BETWEEN SQUARE ROOT BOUNDARIES

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Abstract: Several asymptotic formulas for probabilities of the Wiener process to stay between different square root boundaries are known after papers of Breiman (1965), Sato (1977), Novikov (1979, 1981), Gärtner (1982), Uchiyama (1980), Greenwood and Perkins (1983). In the present work we investigate accuracy of these approximations. We present several general estimates in the case of two boundaries. In particular, these estimates contain the ones obtained earlier by Uchiyama in the limiting case of one boundary.

Keywords: Brownian motion, Wiener process, square root boundaries, exit time, accuracy of approximation

1 Introduction and main results

1.1. Introduction. For a standard Brownian motion $B_t = B(t)$, defined for all $t \geq 0$, introduce into consideration the following stopping time:

$$\tau_{c_1,c_2} := \inf\{t \geq 1 : B_t \notin (c_1\sqrt{t}, c_2\sqrt{t})\} = \inf\{t \geq 1 : U_t \notin (c_1, c_2)\},$$

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where $U_t = U(t) := B(t)/\sqrt{t}$ for $t > 0$ and $c_1 < c_2$. Our main aims are to approximate, for large $T > 1$, the probability

$$\begin{aligned}\mathbf{P}(\tau_{c_1,c_2} > T | B_1 = a) &= \mathbf{P}(\tau_{c_1,c_2} > T | U_1 = a)\\ &= \mathbf{P}(c_1 < U_t < c_2 \ \forall t \in [1, T] | U_1 = a)\end{aligned}$$

and the more general probability

$$P_{c_1,c_2}(a, T, y) = \mathbf{P}(\tau_{c_1,c_2} > T \quad \text{and} \quad U_T \le y | U_1 = a), \quad T > 1, \tag{1}$$

for all parameters a, c_1, c_2, y satisfying the following relations:

$$-\infty < c_1 < c_2 \le \infty, \quad a, y < \infty \quad \text{and} \quad c_1 \le a, y \le c_2. \tag{2}$$

Denote by $\mathcal{D}$ the set of all values (a, c_1, c_2, y) which satisfy conditions from (2). We fix the numbers c_1 and c_2 and we often omit the dependence of our notations on c_1, c_2. We everywhere suppose that $T > 1$.

The study of first-passage times over one of the two square root boundaries for the Brownian motion was initiated by Breiman [1]. After that several results in exit problems with one-sided boundaries were obtained by Sato [8], Novikov [6, 7], Gärtner [4], Uchiyama [9]. For more detailed history see also [2]. All these results were summarized by Greenwood and Perkins [5] in their Lemma 3. We recall the corresponding assertions below in our Lemmas 1 and 5. In particular, for all $c_1 < c_2$ there exist functions

$$\lambda(c_1, c_2) > 0 \qquad \text{and} \quad \psi_{c_1,c_2}(x) > 0 \quad \forall x \in (c_1, c_2) \tag{3}$$

such that

$$T^{\lambda(c_1,c_2)} P_{c_1,c_2}(a, T, y) \to \Psi_{c_1,c_2}(a, y) \ = \psi_{c_1,c_2}(a)\theta_{c_1,c_2}(y) \quad \text{on} \ \ \mathcal{D} \tag{4}$$

as $T \to \infty$, where for all $y \in (c_1, c_2)$

$$\theta_{c_1,c_2}(c_1) = 0 < \theta_{c_1,c_2}(y) := \int_{c_1}^{y} \frac{\psi_{c_1,c_2}(x)}{||\psi_{c_1,c_2}||^2} e^{-x^2/2} dx < \theta_{c_1,c_2}(c_2) < \infty \tag{5}$$

and $\quad 0 < ||\psi_{c_1,c_2}||^2 := \int_{c_1}^{c_2} \psi^2_{c_1,c_2}(x) e^{-x^2/2} dx < \infty$.

Unfortunately, the explicit forms of functions $\lambda(c_1, c_2)$ and $\psi_{c_1,c_2}(x)$ are known only in exceptional cases. For example, Breiman [1] pointed out that

$$\lambda(-1, 1) = 1 \quad \text{and} \quad \lambda(-c, c) = 2 \quad \text{for} \quad c^2 = 3 - \sqrt{6}.$$

Several authors (see, for example, [3]) also noted that

$$\begin{aligned}\lambda(0, \infty) = 1/2 \qquad &\text{and} \qquad \psi_{0,\infty}(x) = \sqrt{2/\pi}x\\ \text{with} \qquad \theta_{0,\infty}(x) = 1 - e^{-x^2/2} \qquad &\text{for} \quad x \ge 0,\end{aligned}$$

where $||\psi_{0,\infty}|| = \sqrt[4]{2/\pi}$.

1.2. Statement of the problem. There arises a natural task to obtain a rate of convergence in (4), i.e. to find estimates for the difference

$$\Delta_{c_1,c_2}(a,T,y) := T^{\lambda(c_1,c_2)}P_{c_1,c_2}(a,T,y) - \Psi_{c_1,c_2}(a,y) \quad \text{on} \quad \mathcal{D}, \tag{6}$$

when $T > 1$. In the particular case, when $c_2 = y = \infty$, such estimate was obtained by Uchiyama [9]. In his Theorem 1.1 for all $c := c_1 \in \mathbb{R}$ he found a remarkable inequality:

$$\begin{aligned} \left|\Delta_{c,\infty}(a,T,\infty)\right| &= \left|T^{\lambda(c,\infty)}\mathbf{P}(\tau_{c,\infty} > T | U_1 = a) - \psi_{c,\infty}(a)\theta_{c,\infty}(\infty)\right| \qquad (7)\\ &< \frac{C_1(\varepsilon,c)}{T^{\varkappa(c)}}\exp(\varepsilon x^2) \quad \text{for any} \quad \varepsilon > 0 \quad \text{and} \quad T > T_1(\varepsilon,c) > 1, \end{aligned}$$

where the constants $C_1(\varepsilon,c)$ and $T_1(\varepsilon,c)$ depend only on the chosen numbers $\varepsilon > 0$ and $c \in \mathbb{R}$ (for the implicit definitions of these constants see formula (1.10) in [9]); in addition

$$\text{for all } c \in \mathbb{R} \text{ the function } \varkappa(c) \text{ is continuous and } \varkappa(c) > 1/2. \tag{8}$$

Our aim is to improve and generalize the estimate (7) of Uchiyama.

1.3. Main estimates. Below we consider only parameters from (2).

Theorem 1. *There exists a number $\varkappa(c_1,c_2) > 0$ such that for any function $u_a > 1$ we have on $\mathcal{D}$ that*

$$\left|\Delta_{c_1,c_2}(a,T,y)\right| \le \sqrt[4]{e(1+8\Lambda(c_1,c_2))}\frac{u_a^{\Lambda(c_1,c_2)}}{T^{\varkappa(c_1,c_2)}}\sqrt[4]{\frac{u_a^2}{u_a^2-1}}\exp\left(\frac{a^2}{2(u_a+1)}\right) \tag{9}$$

for all $T > 1$, where $\Lambda(c_1,c_2) := \lambda(c_1,c_2) + \varkappa(c_1,c_2)$.

Now consider several particular cases.

Corollary 1. *Let $c_2 < \infty$. Then from (9) with $u_a = 2$ we immediately have*

$$\forall\, T > 1 \quad \left|\Delta_{c_1,c_2}(a,T,y)\right| \le \frac{C_2(c_1,c_2)}{T^{\varkappa(c_1,c_2)}} \quad \text{on} \quad \mathcal{D}, \tag{10}$$

where

$$C_2(c_1,c_2) := \sqrt[4]{4e(1+8\Lambda(c_1,c_2))}2^{\Lambda(c_1,c_2)}\exp\left(\frac{\max\{c_1^2,c_2^2\}}{6}\right) < \infty.$$

Unfortunately, we cannot find the optimal function $u_a > 1$ which minimize the right-hand side in (9). Instead, we present several simple approximations to such functions. It is clear, that estimate (10) is bad for large values of $|c_1|$, $|c_2|$, or $|a|$; for example, $C_2(c_1,\infty) = \infty$. To avoid this problem, instead of the simple $u_a = 2$ we should use a function u_a which depend on a and $\Lambda(c_1,c_2)$, especially for large values of $|a|$ and $\Lambda(c_1,c_2)$. Such appropriate functions u_a will be found in subsection 2.7; as a result, we obtain the following sufficiently sharp and general estimate.

Corollary 2. *For any numbers $b_a > 0$ and $T > 1$*

$$\left|\Delta_{c_1,c_2}(a,T,y)\right| \le \sqrt{e^{1+1/b_a}(1+8\Lambda(c_1,c_2))}\,\frac{(1+b_a a^2)^{\Lambda(c_1,c_2)}}{T^{\varkappa(c_1,c_2)}} \quad \text{on } \mathcal{D}. \tag{11}$$

In particular, in the simplest case, when $b_a = 1$,

$$\forall T > 1 \quad \left|\Delta_{c_1,c_2}(a,T,y)\right| \le e\sqrt{1+8\Lambda(c_1,c_2)}\,\frac{(1+a^2)^{\Lambda(c_1,c_2)}}{T^{\varkappa(c_1,c_2)}} \quad \text{on } \mathcal{D}. \tag{12}$$

On the other hand, with $b_a = 1+\varepsilon/\Lambda(c_1,c_2)$ we find from (11) the next two interesting estimates.

Corollary 3. *For any $\varepsilon > 0$ and $T > 1$*

$$\begin{aligned}\left|\Delta_{c_1,c_2}(a,T,y)\right| &\le \frac{C_3(\varepsilon,\Lambda(c_1,c_2))}{T^{\varkappa(c_1,c_2)}}\left(1+\frac{\varepsilon a^2}{\Lambda(c_1,c_2)}\right)^{\Lambda(c_1,c_2)} &(13)\\ &\le \frac{C_3(\varepsilon,\Lambda(c_1,c_2))}{T^{\varkappa(c_1,c_2)}}\exp\left(\varepsilon a^2\right) \qquad \text{on } \mathcal{D}, &(14)\end{aligned}$$

where

$$C_3(\varepsilon,\Lambda) := \sqrt{e(1+8\Lambda)}\exp\left(\frac{\Lambda}{2\varepsilon}\right) < \infty. \tag{15}$$

Note that if $|a| \ge 2$ and $\Lambda(c_1,c_2) \ge e$ then estimate (13) with $\varepsilon = 1/2$ is sharper than the simple inequality (12). Observe also that in (11), (12), and (15) we have the coefficient $\sqrt{1+8\Lambda}$ instead of $\sqrt[4]{1+8\Lambda}$ in the other places of the paper (see the proof of Corollary 2 for the reason).

Remark 1. *If $c_2 = \infty$ then $\Lambda(c_1,\infty) < \infty$ and $\varkappa(c_1,\infty) = \varkappa(c_1)$, where the function $\varkappa(\cdot)$ satisfies conditions (8). In this case inequalities (12) and (13) are sharper than estimate (7) of Uchiyama, at least for sufficiently large $|a|$. In addition, rough inequality (14) is more convenient than (7) because it takes place for all $T > 1$ with explicit constant C_3.*

Our considerations show that sharper estimates (12) and (13) would allow to simplify and shorten several proofs in [2] and [5].

1.4. Key estimates.

Introduce a function:

$$\varphi_v(y) := \sqrt[4]{\frac{v^2}{v^2-1}}\exp\left(\frac{y^2}{2(v+1)}\right), \quad v > 1, \quad y \in \mathbb{R}. \tag{16}$$

Theorem 2. *Let a real number T and functions u_a and v_y be such that*

$$u_a > 1, \quad v_y > 1 \quad \text{and} \quad T \ge u_a v_y \quad \text{on } \mathcal{D}. \tag{17}$$

Then we have the following estimate:

$$\left|\Delta_{c_1,c_2}(a,T,y)\right| \le \frac{u_a^{\Lambda(c_1,c_2)}\varphi_{u_a}(a)}{\sqrt{2\pi}T^{\varkappa(c_1,c_2)}}\int_{c_1}^{y} v_x^{\Lambda(c_1,c_2)}\varphi_{v_x}(x)e^{-x^2/2}dx \quad \text{on } \mathcal{D}, \tag{18}$$

where the numbers $\Lambda(c_1,c_2) > \varkappa(c_1,c_2) > 0$ were introduced in Theorem 1.

The following rough inequalities may be useful in the case of small values of T.

Theorem 3. *With the numbers $\lambda(c_1, c_2)$ defined in (4)*

$$T^{\lambda(c_1,c_2)} \geq \Delta_{c_1,c_2}(a,T,y) \geq -\Psi_{c_1,c_2}(a,y) \qquad \text{on } \mathcal{D}. \tag{19}$$

In addition, for all functions u_a and v_y such that

$$u_a > 1 \quad \text{and} \quad v_y > 1 \quad \text{on } \mathcal{D} \tag{20}$$

we have the next estimate:

$$0 \leq \Psi_{c_1,c_2}(a,y) \leq \frac{u_a^{\lambda(c_1,c_2)}\varphi_{u_a}(a)}{\sqrt{2\pi}} \int_{c_1}^{y} v_x^{\lambda(c_1,c_2)}\varphi_{v_x}(x)e^{-x^2/2}dx \quad \text{on } \mathcal{D}. \tag{21}$$

Remark 2. *The limiting function Ψ_{c_1,c_2}, introduced in (4), always exists and is uniquely defined on $\mathcal{D}$. But the function ψ_{c_1,c_2}, used in all our results, may be defined only up to a positive constant multiplier. For example, we may put*

$$\psi_{c_1,c_2}(a) = C_{c_1,c_2}\Psi_{c_1,c_2}(a,c_2) \quad \text{on } \mathcal{D} \tag{22}$$

with an arbitrary constant C_{c_1,c_2} dependent only on c_1 and c_2.

These facts will be established in Lemma 5 below.

2 Proofs

2.1. Main representations. Introduce into consideration an Ornstein-Uhlenbeck process

$$\omega_t := U(e^t) = e^{-t/2}B(e^t) \quad \text{for} \quad t \geq 0, \quad \text{and let} \quad s := \log T > 0. \tag{23}$$

It is well-known that there exists a function $q_{c_1,c_2}(\cdot,\cdot,\cdot)$ such that in the domain $\mathcal{D}$ for all measurable $A \subset [c_1, c_2]$ and $\mu(y) := e^{-y^2/2}$

$$\mathbf{P}(c_1 < \omega_t < c_2 \ \forall t \in [0,s] \ \text{ and } \omega_s \in A | \omega_0 = a) = \int_A q_{c_1,c_2}(a, e^s, y)\mu(y)dy. \tag{24}$$

This is the probability that the Ornstein-Uhlenbeck process with the absorbing barriers at positions $c_1 < c_2$ is not absorbed until the time $s = \log T$. Here the function $q_{c_1,c_2}(\cdot,\cdot,\cdot)$ has several remarkable properties which were used, for example, in [1] and [9]. The next assertion follows from more general Proposition 2 and Lemma 3 in [5]. See also [9] for more details when $c_2 = \infty$.

Lemma 1. *Let the real numbers a, c_1, c_2, y satisfy conditions (2). Then the function $q_{c_1,c_2}(\cdot,\cdot,\cdot)$, defined in (24), has the following properties:*

(A) *for each fixed $T = e^s > 1$ we have the next representation:*

$$q_{c_1,c_2}(a,T,y) = \sum_{k=0}^{\infty} e^{-\lambda_k s}\phi_k(a)\phi_k(y) = \sum_{k=0}^{\infty} \frac{\phi_k(a)\phi_k(y)}{T^{\lambda_k}} \tag{25}$$

for some functions $\{\phi_k(\cdot) = \phi_k(\cdot, c_1, c_2)\}$ *and numbers* $\{\lambda_k = \lambda_k(c_1, c_2) > 0\}$ *generated by the Ornstein-Uhlenbeck process, where the series converges absolutely, uniformly for* (a, y) *in compact subsets of* $[c_1, c_2]^2$*. The convergence also holds in the space of all square-summable (with respect to the measure* $\mu(a)\mu(y)dady$*) functions in the plane domain* $c_1 < a, y < c_2$*.*

(B) *The function* $\lambda_0(c_1, c_2)$ *is strictly positive, jointly continuous on the set*

$$C := \{(c_1, c_2) : -\infty < c_1 < c_2 \leq \infty\};$$

it is also strictly increasing in $c_l \in (-\infty, c_2]$*, and strictly decreasing in* $c_2 \in [c_1, \infty]$*. In addition,*

$$\lim_{(c_1,c_2)\to(-\infty,\infty)} \lambda_0(c_1, c_2) = 0 \quad \textit{and} \quad \lim_{(c_1,c_2)\to(0,0)} \lambda_0(c_1, c_2) = \infty.$$

(C) *For each* $k > 0$ *and all* $(c_1, c_2) \in C$

$$0 < \lambda_0(c_1, c_2) < \cdots < \lambda_k(c_1, c_2) < \lambda_{k+1}(c_1, c_2) < \infty.$$

Moreover, all functions $\lambda_k(c_1, \infty)$ *are continuous on* $\mathbb{R}$ *with*

$$\lambda_{k+1}(c_1, \infty) - \lambda_k(c_1, \infty) > 1/2 \quad \textit{for all} \quad k = 0, 1, \ldots \quad \textit{and} \ \ c_1 \in \mathbb{R}.$$

(D) *The function* $\phi_0(\cdot) = \phi_0(\cdot, c_1, c_2)$ *is positive on* (c_1, c_2)*, moreover*

$$\forall \varepsilon > 0 \quad \inf\{\phi_0(x, c_1, c_2) : c_1 - \varepsilon < x < c_2 - \varepsilon\} > 0.$$

In addition, the function $\phi_0(\cdot, c_1, \infty)$ *is continuous on* $[c_1, \infty)$*.*

(E) *functions* $\{\phi_k(\cdot) = \phi_k(\cdot, c_1, c_2)\}$ *form a complete orthonormal system in the space* $L^2_{c_1,c_2}$ *of all square-summable (with respect to the measure* $\mu(y)dy$*) functions on* (c_1, c_2)*; in particular, for all* $j > k \geq 0$

$$\int_{c_1}^{c_2} \phi_k^2(y, c_1, c_2)\mu(y)dy = 1 \quad \textit{and} \quad \int_{c_1}^{c_2} \phi_k(y, c_1, c_2)\phi_j(y, c_1, c_2)\mu(y)dy = 0.$$

2.2. Corollaries from Lemma 1. Under conditions (2) consider the function:

$$r_{c_1,c_2}(a, T, y) := T^{\lambda_0} q_{c_1,c_2}(a, T, y) - \phi_0(a)\phi_0(y) = T^{\lambda_0} \sum_{k=1}^{\infty} \frac{\phi_k(a)\phi_k(y)}{T^{\lambda_k}} \tag{26}$$

with the same functions $\{\phi_k(\cdot)\}$ and numbers $\{\lambda_k\}$ as in (25).

Lemma 2. *Suppose that conditions* (2) *and* (17) *are fulfilled. Then*

$$T^{2(\lambda_1-\lambda_0)} r^2_{c_1,c_2}(a, T, y) \leq (u_a v_y)^{2(\lambda_1-\lambda_0)} r_{c_1,c_2}(a, u_a^2, a) r_{c_1,c_2}(y, v_y^2, y) \tag{27}$$

$$\leq (u_a v_y)^{2\lambda_1} q_{c_1,c_2}(a, u_a^2, a) q_{c_1,c_2}(y, v_y^2, y). \tag{28}$$

Proof. Note first of all that by Schwartz inequality

$$\Sigma := \sum_{k=1}^{\infty} \frac{|\phi_k(a)|}{u_a^{\lambda_k}} \frac{|\phi_k(y)|}{v_y^{\lambda_k}} \le \sqrt{\sum_{k=1}^{\infty} \frac{\phi_k^2(a)}{u_a^{2\lambda_k}}} \sqrt{\sum_{k=1}^{\infty} \frac{\phi_k^2(y)}{v_y^{2\lambda_k}}} = \sqrt{\frac{r(a, u_a^2, a)}{u_a^{2\lambda_0}}} \sqrt{\frac{r(y, v_y^2, y)}{v_y^{2\lambda_0}}}. \tag{29}$$

Here and below we use simplified notation $r(\cdot,\cdot,\cdot)$ instead of $r_{c_1,c_2}(\cdot,\cdot,\cdot)$.

On the other hand, we have from assumptions (17) that for all $k \ge 1$

$$\frac{1}{T^{\lambda_k}} = \frac{1}{T^{\lambda_1}} \frac{1}{T^{\lambda_k - \lambda_1}} \le \frac{1}{T^{\lambda_1}} \frac{1}{(u_a v_y)^{\lambda_k - \lambda_1}} = \frac{(u_a v_y)^{\lambda_1}}{T^{\lambda_1}} \frac{1}{(u_a v_y)^{\lambda_k}}.$$

This elementary relation together with the definition of $r(a, T, y)$ in (26) imply that

$$\frac{|r(a,T,y)|}{T^{\lambda_0}} \le \sum_{k=1}^{\infty} \frac{|\phi_k(a)\phi_k(y)|}{T^{\lambda_k}} \le \frac{(u_a v_y)^{\lambda_1}}{T^{\lambda_1}} \sum_{k=1}^{\infty} \frac{|\phi_k(a)\phi_k(y)|}{(u_a v_y)^{\lambda_k}} = \frac{(u_a v_y)^{\lambda_1}}{T^{\lambda_1}} \Sigma. \tag{30}$$

Substituting now (29) into (30) we arrive at (27).

Next, it follows from representations (25) and (26) that

$$\frac{r_{c_1,c_2}(y,T,y)}{T^{\lambda_0}} = \sum_{k=1}^{\infty} \frac{\phi_k^2(y)}{T^{\lambda_k}} \le \sum_{k=0}^{\infty} \frac{\phi_k^2(y)}{T^{\lambda_k}} = q_{c_1,c_2}(y,T,y).$$

Using this inequality with $T = v_y^2$ and with $T = u_a^2$, when $y = a$, we obtain (28) as a consequence of (27). □

For all $x \in \mathbb{R}$ and $t > 0$ denote by $\varphi(t,x) := \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)$ the density of B_t.

Lemma 3. *Under conditions* (2)

$$q_{c_1,c_2}(a,T,y) \le \varphi(1 - 1/T, y - a/\sqrt{T})/\mu(y). \tag{31}$$

In particular, when $a = y$ *and* $t = v^2$*,*

$$\frac{\phi_0^2(y)}{v^{2\lambda_0}} \le q_{c_1,c_2}(y, v^2, y) \le \frac{\varphi(1 - 1/v^2, y - y/v)}{\mu(y)} = \frac{\varphi_v^2(y)}{\sqrt{2\pi}}, \tag{32}$$

where $\varphi_v(y)$ *was defined in* (16).

Proof. Note that $\varphi(1 - 1/T, \cdot - a/\sqrt{T})$ is the density of $U_T = B_T/\sqrt{T}$ under condition that $B_1 = a$. Hence, under assumptions (2), we have from (24) and (23) that

$$\int_A q_{c_1,c_2}(a,T,y)\mu(y)dy \le \mathbf{P}(\omega_{\log T} \in A | \omega_0 = a) = \mathbf{P}(U_T \in A | B_1 = a) = \int_A \varphi(1 - 1/T, y - a/\sqrt{T})dy, \tag{33}$$

and, as a result, (31) follows for each $y \in (c_1, c_2)$ because (33) holds for all sets $A \subset (c_1, c_2)$.

Next, for $v > 1$

$$\frac{y^2}{2} - \frac{(y - y/v)^2}{2(1 - 1/v^2)} = \frac{y^2}{2} - \frac{y^2(v-1)^2}{2(v^2-1)} = \frac{y^2(v+1)}{2(v+1)} - \frac{y^2(v-1)}{2(v+1)} = \frac{y^2}{v+1}.$$

Hence, in this case

$$\frac{\varphi(1 - 1/v^2, y - y/v)}{\mu(y)} = \frac{1}{\sqrt{2\pi(1 - 1/v^2)}} \exp\left(\frac{y^2}{v+1}\right) = \frac{1}{\sqrt{2\pi}} \varphi_v^2(y).$$

So, (32) is also proved with $\varphi_v(y)$ introduced in (16). □

2.3. Proof of Theorem 2 .

Below in the paper we use the next notations:

$$\lambda := \lambda(c_1, c_2) := \lambda_0(c_1, c_2) = \lambda_0 < \Lambda := \Lambda(c_1, c_2) := \lambda_1(c_1, c_2) = \lambda_1, \quad (34)$$

$$\varkappa := \varkappa(c_1, c_2) := \lambda_1 - \lambda_0 > 0, \quad \theta_0(y, c_1, c_2) := \int_{c_1}^{y} \phi_0(x, c_1, c_2)\mu(x)dx.$$

Note also that by (23), (24), and (1)

$$P_{c_1,c_2}(a, T, y) = \int_{c_1}^{y} q_{c_1,c_2}(a, T, x)\mu(x)dx \qquad \text{on} \ \ \mathcal{D}. \quad (35)$$

Lemma 4. *Suppose that the following equality*

$$\Psi_{c_1,c_2}(a, y) \ = \psi_{c_1,c_2}(a)\theta_{c_1,c_2}(y) = \Psi_0(a, y, , c_1, c_2) := \phi_0(a, c_1, c_2)\theta_0(y, c_1, c_2) \quad (36)$$

holds in domain $\mathcal{D}$. Then estimate (18) and convergence (4) take place. In particular, condition (36) is obviously fulfilled when $\psi_{c_1,c_2}(a) = \phi_0(a, c_1, c_2)$.

Proof. If condition (36) is true, then we see from (6), (26), and (35) that

$$\Delta_{c_1,c_2}(a, T, y) = \int_{c_1}^{y} r_{c_1,c_2}(a, T, x)\mu(x)dx. \quad (37)$$

Using now (28), (32), and (34) we get

$$|r_{c_1,c_2}(a, T, y)| \le \frac{(u_a v_y)^{\lambda_1}}{T^{\lambda_1 - \lambda_0}} \sqrt{q_{c_1,c_2}(a, u_a^2, a) q_{c_1,c_2}(y, v_y^2, y)}$$
$$\le \frac{(u_a v_y)^{\lambda_1}}{T^{\lambda_1 - \lambda_0}} \frac{\varphi_{u_a}(a)\varphi_{v_y}(y)}{\sqrt{2\pi}} = \frac{(u_a v_y)^{\Lambda(c_1,c_2)}}{T^{\varkappa(c_1,c_2)}} \frac{\varphi_{u_a}(a)\varphi_{v_y}(y)}{\sqrt{2\pi}}.$$

Plugging the last inequality into (37) we arrive at (18).

At last, convergence (4) follows from estimate (18) with $u_a = v_y = 2$. □

Lemma 5. *Convergence (4) takes place if and only if condition (36) is fulfilled. In addition, (36) holds if and only if there exists a constant $C > 0$, independent of a and y, such that*

$$\psi_{c_1,c_2}(a) = C\phi_0(a, c_1, c_2) \qquad \text{on} \ \ \mathcal{D}. \quad (38)$$

In particular, formula (22) with $C_{c_1,c_2} = C/\theta_0(y, c_1, c_2)$ immediately follows from (38) and (36).

Proof. When $\psi_{c_1,c_2} = \phi_0$, convergence (4) takes place by Lemma 4 with $\Psi_{c_1,c_2}(a,y) = \Psi_0(a,y,c_1,c_2)$. So, this equality holds for all functions ψ_{c_1,c_2} which is possible to use in (4). But condition $\Psi_{c_1,c_2}(a,y) = \Psi_0(a,y,c_1,c_2)$ coincides with (36).

Next, assume that condition (36) is fulfilled. Recall that $||\phi_0||^2 := 1$ as it follows from assertion (E) of Lemma 1. Hence in this case we have from (5) that

$$||\psi_{c_1,c_2}||^2 = C^2 > 0 \quad \text{and} \quad \theta_{c_1,c_2}(y) := \int_{c_1}^{y} \frac{C\phi_0(x,c_1,c_2)}{C^2}\mu(x)dx.$$

Substituting these equalities into (36) we see that condition (36) holds because $C^2/C^2 = 1$ and the product $\psi_{c_1,c_2}(a)\theta_{c_1,c_2}(y)$ is independent of $C > 0$.

On the other hand, when condition (36) holds with some possible function ψ_{c_1,c_2}, we have from (36) with $y = c_2$ that identity (38) takes place with

$$C = \int_{c_1}^{c_2} \phi_0(x,c_1,c_2)\mu(x)dx \Big/ \theta_{c_1,c_2}(c_2) > 0.$$

□

So, there are no cases in Theorem 2 when condition (36) does not hold. Hence, Theorem 2 is proved in Lemma 4 in all cases.

2.4. Proof of Theorem 3. From (3) and (32) with $v = v_y$ and $v = u_a$ we immediately have that

$$0 \le \phi_0(y) \le \frac{v_y^{\lambda}\varphi_{v_y}(y)}{\sqrt[4]{2\pi}} \quad \text{and} \quad 0 \le \phi_0(a) \le \frac{u_a^{\lambda}\varphi_{u_a}(a)}{\sqrt[4]{2\pi}} \qquad \text{on } \mathcal{D}. \tag{39}$$

Now, by definition (34),

$$0 \le \theta_0(y,c_1,c_2) = \int_{c_1}^{y} \phi_0(x)\mu(x)dx \le \int_{c_1}^{y} \frac{v_x^{\lambda}\varphi_{v_x}(x)}{\sqrt[4]{2\pi}}\mu(x)dx. \tag{40}$$

Substituting (39) and (40) into the representation (36) for $\Psi_{c_1,c_2}(a,y)$ we arrive at (21).

At last, (19) is evident by (6), because the probability $P_{c_1,c_2}(a,T,y) \le 1$. So, both assertions of Theorem 3 are proved.

2.5. Two auxiliary lemmas. First of all note that for $v > 1$

$$\beta_2(v) := \sqrt[4]{1 - 1/v^2} = \sqrt[4]{\frac{v^2}{v^2-1}} = \sqrt[4]{\frac{v^2}{(v-1)(v+1)}} < \beta_1(v) := \sqrt[4]{\frac{v+1}{v-1}}. \tag{41}$$

Lemma 6. *For all* $y \ge c_1$

$$\overline{\theta}_v(y) := \int_{c_1}^{y} \varphi_v(x)\mu(x)dx \le \sqrt{2\pi}\beta_1(v), \quad v > 1. \tag{42}$$

Proof. From definition (16) we see that

$$\varphi_v(y)\mu(y) = \beta_2(v)\exp\left(\frac{y^2}{2(v+1)} - \frac{y^2}{2}\right) = \beta_2(v)\exp\left(-\frac{y^2}{2\sigma^2(v)}\right), \quad v > 1.$$

with $\sigma^2(v) := (v+1)/v$. Hence

$$\overline{\theta}_v(y) = \int_{c_1}^{y} \varphi_v(x)\mu(x)dx \le \beta_2(v)\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\sigma^2(v)}\right)dx = \sqrt{2\pi}\beta_2(v)\sigma(v) \tag{43}$$

for $v > 1$, where

$$\beta_2(v)\sigma(v) = \sqrt[4]{\frac{v^2}{v^2-1}} = \sqrt[4]{\frac{v^2}{(v-1)(v+1)}\cdot\frac{(v+1)^2}{v^2}} = \sqrt[4]{\frac{v+1}{v-1}} = \beta_1(v).$$

Thus, (42) follows from (43). □

For $i = 1, 2$ introduce functions:

$$f_1(v) := \beta_1(v)v^{\Lambda} \quad \text{and} \quad f_2(v,y) := \beta_2(v)v^{\Lambda}\varphi_v(y), \quad v > 1, \quad y \in \mathbb{R}, \tag{44}$$

where the value $\Lambda = \lambda_1$ was defined in (34). Note that

$$0 < \varkappa < \Lambda < \infty \quad \text{and} \quad 1 < v_* := 1 + \frac{1}{4\Lambda} < 1 + \frac{1}{\varkappa} < \infty.$$

Lemma 7. *For all $y \in \mathbb{R}$ and $K \ge 1$*

$$f_2(v,y) \le \sqrt[4]{e(1+8\Lambda)}K^{\Lambda}\exp\left(\frac{y^2}{2v}\right) \quad \textit{when} \quad v_* \le v \le v_* K. \tag{45}$$

In addition, for all $y \ge c_1$

$$v_*^{\Lambda}\overline{\theta}_{v_*}(y) \le v_*^{\Lambda}\sqrt{2\pi}\beta_1(v_*) = \sqrt{2\pi}f_1(v_*) \le \sqrt{2\pi}\sqrt[4]{e(1+8\Lambda)}. \tag{46}$$

Proof. From definitions (41) and conditions (45) on number v we have:

$$\beta_2^4(v) < \beta_1^4(v) = 1 + \frac{2}{v-1} \le 1 + \frac{2}{v_*-1} = 1 + 8\Lambda,$$

$$v^{\Lambda} \le v_*^{\Lambda}K^{\Lambda} = \left(1 + \frac{1}{4\Lambda}\right)^{\Lambda}K^{\Lambda} \le e^{1/4}K^{\Lambda} = \sqrt[4]{e}K^{\Lambda}.$$

Substituting this estimates into definition (44) of functions f_i, we arrive at (45). When $K = 1$, these arguments and (42) also imply (46). □

2.6. Proof of Theorem 1 . We are going to show that under assumption (**??**)

$$\forall T > 1 \;\; \forall u_a > 1 \quad \left|\Delta_{c_1,c_2}(a,T,y)\right| \le \frac{f_2(u_a,a)f_1(v_*)}{T^{\varkappa}} \qquad \text{on} \;\; \mathcal{D}. \tag{47}$$

First of all we apply Theorem 2 with $v_y = v_*$. In this case the main condition (17) of this theorem has the form:

$$u_a > 1 \qquad \text{and} \qquad T \ge u_a v_*, \tag{48}$$

and assertion (18) may be rewritten in the following way:

$$\left|\Delta_{c_1,c_2}(a,T,y)\right| \le \frac{u_a^\Lambda \varphi_{u_a}(a)}{\sqrt{2\pi}T^\varkappa} v_*^\Lambda \theta_{v_*}(y) = \frac{f_2(u_a,a)}{\sqrt{2\pi}T^\varkappa} v_*^\Lambda \theta_{v_*}(y) \quad \text{on } \mathcal{D}, \tag{49}$$

where we used simplified notations, introduced in (16), (42), and (44). But now (47) immediately follows from (49) and (46).

Thus, we proved (47) when assumption (48) is true. Now suppose that (48) is not fulfilled and assume instead that the next condition takes place:

$$u_a > 1 \qquad \text{and} \qquad 1 < T \le u_a v_*. \tag{50}$$

Now we will apply Theorem 3. From the first inequality in (19), using (50), we get:

$$\Delta_{c_1,c_2}(a,T,y) \le T^\lambda = \frac{T^\Lambda}{T^\varkappa} \le \frac{(u_a v_*)^\Lambda}{T^\varkappa} \tag{51}$$
$$\le \frac{(u_a v_*)^\Lambda}{T^\varkappa} \varphi_{u_a}(a)\beta_1(v_*) = \frac{f_2(u_a,a)f_1(v_*)}{T^\varkappa}$$

on $\mathcal{D}$, because $\varphi_u(a) > 1$ and $\beta_1(v) > 1$.

On the another hand, under assumption (50) we find from (19) and (21) that

$$-\Delta_{c_1,c_2}(a,T,y) \le \Psi_{c_1,c_2}(a,y) \le \frac{(u_a v_*)^\varkappa}{T^\varkappa}\Psi_{c_1,c_2}(a,y)$$
$$\le \frac{(u_a v_*)^\varkappa}{T^\varkappa}\frac{u_a^\lambda \varphi_{u_a}(a)}{\sqrt{2\pi}} v_*^\lambda \overline{\theta}_{v_*}(y) = \frac{u_a^\Lambda \varphi_{u_a}(a)}{T^\varkappa\sqrt{2\pi}} v_*^\Lambda \overline{\theta}_{v_*}(y) \quad \text{on } \mathcal{D}.$$

Using now notations from (44) and estimate (46) we obtain:

$$-\Delta_{c_1,c_2}(a,T,y) \le \frac{f_2(u_a,a)}{T^\varkappa\sqrt{2\pi}} v_*^\Lambda \overline{\theta}_{v_*}(y) \le \frac{f_2(u_a,a)}{T^\varkappa} f_1(v_*) \quad \text{on } \mathcal{D}.$$

Thus, it follows from the last inequality and (51) that estimate (47) is true under assumption (50).

So, (47) is proved in all cases. But the desired estimate (9) in Theorem 1 follows immediately from (47) and (46). Hence, Theorem 1 is also proved.

2.7. Proof of Corollary 2. We are going to use inequality (45) for a instead of y and u_a instead of v, where

$$u_a = v_* K, \quad \text{with} \quad K = (1 + b_a a^2),\ b_a > 0.$$

Then $a^2/u_a \le a^2/K \le 1/b_a$. Hence, by (45),

$$f_2(u_a,a) \le \sqrt[4]{e(1+8\Lambda)}(1+b_a a^2)^\Lambda \exp\left(\frac{1}{2b_a}\right) \quad \text{on } \mathcal{D}.$$

Substituting this estimate into (9), we obtain:

$$\begin{aligned}\left|\Delta_{c,\infty}(a,T,y)\right| &\le \sqrt[4]{e(1+8\Lambda)}\,\frac{f_2(u_a,a)}{T^{\varkappa}}\\ &\le \sqrt[4]{e(1+8\Lambda)}\,\frac{\sqrt[4]{e(1+8\Lambda)}(1+b_a a^2)^{\Lambda}}{T^{\varkappa}}\exp\left(\frac{1}{2b_a}\right) \quad \text{on } \mathcal{D}.\end{aligned}$$

It is easy to see that the last inequality coincides with (11).

Thus, Corollary 2 is true. Hence, all the results of the paper are proved.

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