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ON EQUILIBRIUM PROBLEM FOR $3D$ ELASTIC BODY WITH THIN AND VOLUME INCLUSIONS IN NON-COERCIVE CASE

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Abstract: In the paper, we analyze an equilibrium problem for a three-dimensional elastic body with incorporated thin two-dimensional elastic inclusion. The inclusion is delaminated from the surrounding elastic body forming therefore an interfacial crack. To avoid a mutual penetration between the opposite crack faces, the inequality type boundary conditions at the faces are considered. In addition to this, the Neumann type condition is imposed at the external boundary of the elastic body, which implies a non-coercivity of the problem. Assuming that the elasticity tensor depends on a positive parameter in a given subdomain we prove an existence of solutions and investigate the asymptotic behavior of the solutions as this parameter tends to infinity. It is proved that the limit model describes an equilibrium state of the elastic body with the thin and volume inclusions. Necessary and sufficient conditions imposed on the external forces are found, and solution existence for all cases analyzed is proved.

Keywords: elastic body, crack, volume rigid inclusion, Neumann and inequality type boundary conditions, asymptotics of solutions

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1 Introduction

Mathematical modeling of composites leads to the need to analyze boundary value problems in non-smooth domains. In particular, composite structures may contain thin inclusions of various natures: elastic, rigid, semi-rigid, etc. In the cases of delaminations of thin inclusions from the surrounding elastic material we have to analyze boundary value problems with interfacial cracks. From mechanical point of view, boundary conditions of inequality type at the crack faces look more suitable; they allow to avoid a mutual penetration between faces. Last decades, a huge number of works have been published related to the analysis of composites structures with thin and volume inclusions, see, for example, [1-18] and the literature therein. The articles [19-21] are focusing on the volume rigid inclusions.

In the present paper, we consider an equilibrium problem for $3D$ elastic body with incorporated thin elastic inclusion modeled in the frame of the Kirchhoff-Love approach. The inclusion is delaminated from the elastic body. Boundary conditions of inequality type are imposed at the crack faces.
Simultaneously, we consider the Neumann boundary condition at the external boundary of the elastic body which implies a non-coercivity of the problem analyzed. We can refer the reader to the works [22-29] where various non-coercive boundary value problems are investigated.

The structure of the paper is as follows. Section 2 concerns the problem statement and provides both variational and differential approaches. A solution existence is proved. In so doing, necessary and sufficient conditions imposed on the external forces are found. In Section 3, an asymptotic analysis of the solutions is presented. We assume that the elasticity tensor is changing in a given subdomain being proportional to the positive parameter. A passage to the limit is investigated as the parameter tends to infinity. The limit model is analyzed, and the solution existence is proved. The limit model describes an equilibrium state of the elastic body with the thin elastic and volume rigid inclusion.

2 Problem formulation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary Γ, and $\bar{\gamma} \subset \Omega$; γ is a $2D$ domain lying in a plane $x_3 = 0$ with a smooth boundary, see Fig.1. Denote by $n = (n_1, n_2, n_3)$ a unit normal vector to Γ; $\nu = (\nu_1, \nu_2)$ is the unit vector to the boundary $\partial\gamma$ of γ in the plane $x_3 = 0$; $\Omega_\gamma = \Omega \setminus \bar{\gamma}$; $(0, 0, 0) \in \partial\gamma$.

Introduce elasticity tensors $A = \{a_{ijkl}\}, i, j, k, l = 1, 2, 3;$ $B = \{b_{ijkl}\}$, $D = \{d_{ijkl}\}$, $i, j, k, l = 1, 2$, satisfying the symmetry and positive definiteness properties

$$A\xi \cdot \xi \ge c_0|\xi|^2 \ \ \forall \xi = \{\xi_{ij}\},\ c_0 = \text{const} > 0;$$
$$a_{ijkl} = a_{jikl} = a_{klij},\ a_{ijkl} \in L^\infty(\Omega).$$

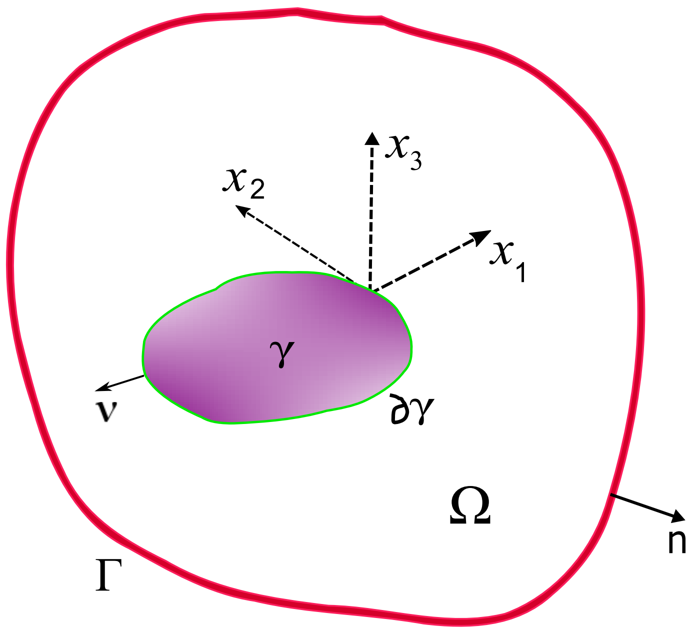

FIG. 1. Elastic body with inclusion γ

All functions with two lower indices are symmetric in those indices. Summation convention over repeated indices is used; functions defined on γ are identified with functions of the variable $(x_1, x_2) \in \gamma$.

For a scalar function w defined in γ we put $\nabla\nabla w = \{w_{,ij}\}, i, j = 1, 2$. If $M = \{M_{ij}\}, i, j = 1, 2$, then $\nabla\nabla M = M_{ij,ij}$. Introduce notations for a bending moment M^ν and a transverse force $T^\nu = T^\nu(M)$,

$$M^\nu = -M_{ij}\nu_j\nu_i;\ T^\nu = -M_{ij,j}\nu_i - M_{ij,k}\tau_k\tau_j\nu_i\ ,\ (\tau_1, \tau_2) = (-\nu_2, \nu_1). \quad (1)$$

The domain Ω_γ corresponds to an elastic body, and γ fits to the thin elastic inclusion.

The formulation of the problem is as follows. We have to find a displacement field $u = (u_1, u_2, u_3)$ of the elastic body and a displacement

$v = (v_1, v_2), w$ of the thin inclusion defined on γ such that

$$-\mathrm{div}\sigma = g,\ \sigma = A\varepsilon(u) \text{ in } \Omega_\gamma, \tag{2}$$

$$-\mathrm{div}N = [(\sigma_{13}, \sigma_{23})],\ \ N = B\varepsilon(v) \text{ on } \gamma, \tag{3}$$

$$-\nabla\nabla M = [\sigma_{33}],\ \ M = -D\nabla\nabla w \text{ on } \gamma, \tag{4}$$

$$\sigma n = 0 \ \text{ on } \Gamma;\ N\nu = 0,\ M^\nu = T^\nu = 0 \text{ on } \partial\gamma, \tag{5}$$

$$u_i^- = v_i,\ i = 1, 2;\ u_3^- = w;\ [u_3] \geq 0 \text{ on } \ \gamma, \tag{6}$$

$$\sigma_{33}^+ \leq 0,\ \sigma_{i3}^+ = 0,\ i = 1, 2;\ [u_3]\sigma_{33}^+ = 0 \text{ on } \gamma, \tag{7}$$

$$\int\limits_{\Omega_\gamma} u = 0,\ \int\limits_{\Omega_\gamma} (u_{i,j} - u_{j,i}) = 0,\ i, j = 1, 2, 3. \tag{8}$$

Here, $N = N(v) = \{N_{ij}\},\ M = M(w) = \{M_{ij}\}, i, j = 1, 2,$ are stress and moment tensors to be defined on γ; $\sigma = \sigma(u) = \{\sigma_{ij}\}, i, j = 1, 2, 3,$ is the stress tensor for the elastic body to be defined in Ω_γ; $\varepsilon(u) = \{\varepsilon_{ij}(u)\},\ \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$ is the strain tensor, $i, j = 1, 2, 3$; $[h] = h^+ - h^-$ is a jump of a function h on γ, where the signs $\pm$ correspond to positive and negative faces of the crack with respect to the normal vector $(0, 0, 1)$. It is assumed that γ is delaminated from the elastic material, thus forming a crack between γ and the elastic body. In particular, the delamination of the elastic inclusion means that a displacement field of the elastic body has a jump on γ. We assume that the delamination takes place at γ^+. In such a case we impose suitable glue conditions at γ^-, see (6). Relations (2) are the equilibrium equation of the elastic body and the constitutive law (Hooke's law), respectively. Relations (3), (4) provide equilibrium and constitutive law equations for the thin inclusion γ being an elastic plate in the frame of Kirchhoff-Love approach. As for boundary conditions (6)-(7), they guarantee a mutual non-penetration between the crack faces $\gamma^\pm$ with zero friction at γ^+. The contact set is unknown a priori in the frame of the model (2)-(8); the approach used corresponds to a free boundary one. The left-hand sides in the first equations of (3) and (4) describe forces acting on γ from the surrounding elastic body. The first boundary condition of (5) corresponds to the Neumann type which implies the non-coercivity of the problem (2)-(7). To obtain a unique solution, we impose integral conditions (8). The function $g \in L^2(\Omega)^3$ represents a given external force acting on the elastic body.

Our aim is to prove a solution existence of the problem (2)-(8) and analyze a passage to a limit with respect to a parameter, to be introduced in Section 3, characterizing the elasticity tensor A.

Consider Sobolev spaces $H^1(\Omega_\gamma)^3,\ H^i(\gamma), i = 1, 2,$

$$U = \{(u, v, w) \in H^1(\Omega_\gamma)^3 \times H^1(\gamma)^2 \times H^2(\gamma) \mid u_i^- = v_i,\ i = 1, 2,\\ u_3^- = w \text{ on } \gamma; u = (u_1, u_2, u_3),\ v = (v_1, v_2)\}$$

and its subspace $U_1 \subset U$,

$$U_1 = \{(u, v, w) \in U \mid \int\limits_{\Omega_\gamma} u = 0, \ \int\limits_{\Omega_\gamma} (u_{i,j} - u_{j,i}) = 0, \quad i, j = 1, 2, 3; \ u = (u_1, u_2, u_3)\}.$$

The norm in the space U is defined as follows

$$\|(u, v, w)\|_U^2 = \|u\|^2_{H^1(\Omega_\gamma)^3} + \|v\|^2_{H^1(\gamma)^2} + \|w\|^2_{H^2(\gamma)}.$$

For any set $G \subset \mathbb{R}^3$, introduce the space of infinitesimal rigid displacements

$$R(G) = \{\rho = (\rho_1, \rho_2, \rho_3) \mid \rho(x) = C + Fx, \ x \in G; \ C = (c^1, c^2, c^3), \quad F = \{f_{ij}\}, f_{ij} = -f_{ji}, \ i, j = 1, 2, 3; \ c^i, f_{ij} \in \mathbb{R}\}$$

and the following spaces

$$L(\gamma) = \{l = (l_{12}, l_3) \mid l_{12} = (l_1, l_2), \ l_1(x) = ax_2 + c^1, \ l_2(x) = -ax_1 + c^2, \quad l_3(x) = a_1x_1 + a_2x_2 + c^3; \ x = (x_1, x_2) \in \gamma; \ a, a_i, c^i \in \mathbb{R}\},$$

$$S = \{(\rho, l_{12}, l_3) \in U \mid \rho \in R(\Omega_\gamma), \ (l_{12}, l_3) \in L(\gamma)\}.$$

Introduce the following inner product in the space U,

$$\langle (u, v, w), (\bar{u}, \bar{v}, \bar{w}) \rangle = \int\limits_{\Omega_\gamma} u_i \int\limits_{\Omega_\gamma} \bar{u}_i + \int\limits_{\Omega_\gamma} u_{i,j}\bar{u}_{i,j} + \quad (9)$$

$$+ \int\limits_\gamma \varepsilon_{kp}(v)\varepsilon_{kp}(\bar{v}) + \int\limits_\gamma w_{,kp}\bar{w}_{,kp}; \ i, j = 1, 2, 3; \ k, p = 1, 2.$$

The inner product (9) induces the norm in the space U being equivalent to the standard norm in this space. We omit the proof of this statement here. Suitable arguments will be presented below where the coercivity of the energy functional E is proved.

Consider the energy functional $E: \ U \to \mathbb{R}$,

$$E(u, v, w) = \frac{1}{2} \int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(u) - \int\limits_{\Omega_\gamma} gu +$$

$$+ \frac{1}{2} \int\limits_\gamma N(v)\varepsilon(v) - \frac{1}{2} \int\limits_\gamma M(w)\nabla\nabla w, \ u = (u_1, u_2, u_3), \ v = (v_1, v_2).$$

For simplicity we write $\sigma(u)\varepsilon(u)$, $N(v)\varepsilon(v)$ instead of $\sigma_{ij}(u)\varepsilon_{ij}(u)$, $N_{ij}(v)\varepsilon_{ij}(v)$.

Introduce next the set of admissible displacements

$$Q = \{(u, v, w) \in U \mid \ [u_3] \geq 0 \text{ on } \ \gamma; \ u = (u_1, u_2, u_3)\},$$

and consider the minimization problem

$$\inf_{(u,v,w)\in Q\cap U_1} E(u,v,w). \tag{10}$$

The problem (10) has a solution (u,v,w) satisfying the variational inequality

$$(u,v,w)\in Q\cap U_1, \tag{11}$$

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u}-u) - \int\limits_{\Omega_\gamma} g(\bar{u}-u) + \tag{12}$$

$$+\int\limits_{\gamma} N(v)\varepsilon(\bar{v}-v) - \int\limits_{\gamma} M(w)\nabla\nabla(\bar{w}-w) \geq 0 \quad \forall(\bar{u},\bar{v},\bar{w})\in Q\cap U_1.$$

To prove the solution existence of the problem (11)-(12), it suffices to check a coercivity of the functional E on the set $Q\cap U_1$ since its weak lower semicontinuity is clear. For $(u,v,w)\in U_1$, we have the following relations with positive constants independent of functions

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(u) \geq c\int\limits_{\Omega_\gamma} \varepsilon(u)\varepsilon(u) \geq c_1\{\int\limits_{\Omega_\gamma} u_{i,j}u_{i,j} + \int\limits_{\Omega_\gamma} u_i \int\limits_{\Omega_\gamma} u_i\} \geq c_2\|u\|^2_{H^1(\Omega_\gamma)^3}. \tag{13}$$

The second inequality in (13) follows from Proposition 1.4 in [12] being valid for the domain Ω_γ with the cut γ. Consequently, we conclude with a parameter $\delta>0$,

$$E(u,v,w) \geq c_3\|u\|^2_{H^1(\Omega_\gamma)^3} - c_4\|u\|_{H^1(\Omega_\gamma)^3} + \frac{1}{2}\int\limits_{\gamma} N(v)\varepsilon(v) - \tag{14}$$

$$-\frac{1}{2}\int\limits_{\gamma} M(w)\nabla\nabla w \pm \delta\int\limits_{\gamma}(v^2+w^2).$$

By imbedding theorems and boundary conditions $u_i^- = v_i$, $i=1,2$, $u_3^- = w$ on γ, it implies for a small δ

$$\frac{c_3}{2}\|u\|^2_{H^1(\Omega_\gamma)^3} - \delta\int\limits_{\gamma}(v^2+w^2) \geq 0.$$

Then

$$\frac{1}{2}\int\limits_{\gamma} N(v)\varepsilon(v) - \frac{1}{2}\int\limits_{\gamma} M(w)\nabla\nabla w + \delta\int\limits_{\gamma}(v^2+w^2) \geq$$

$$\geq c(\|v\|^2_{H^1(\gamma)^2} + \|w\|^2_{H^2(\gamma)}). \tag{15}$$

By (14), (15), the functional E is coercive on the set $Q\cap U_1$, and the problem (11)-(12) has a solution.

In the sequel, we will use the following statement.

Proposition 1. *The space U can be presented as a direct sum of two orthogonal subspaces with respect to the inner product* (9),

$$U = S \oplus U_1.$$

Proof. Take any elements $(\rho, l_{12}, l_3) \in S$ and $(u, v, w) \in U$. Let $\rho(x) = C + Fx,\ x \in \Omega_\gamma;\ C = (c^1, c^2, c^3),\ F = \{f_{ij}\},\ f_{ij} = -f_{ji};\ c^i, f_{ij} \in \mathbb{R},\ i, j = 1, 2, 3$. Then we have

$$\langle (u,v,w), (\rho, l_{12}, l_3)\rangle = c^i |\Omega_\gamma| \int\limits_{\Omega_\gamma} u_i + f_{ij} \int\limits_{\Omega_\gamma} (u_{i,j} - u_{j,i}) +$$

$$+ f_{ij} \Big(\int\limits_{\Omega_\gamma} u_i \int\limits_{\Omega_\gamma} x_j - \int\limits_{\Omega_\gamma} u_j \int\limits_{\Omega_\gamma} x_i \Big);\ u = (u_1, u_2, u_3).$$

Thus, we conclude that sufficient and necessary conditions for the identity $\langle (u,v,w), (\rho, l_{12}, l_3)\rangle = 0\ \ \forall (\rho, l_{12}, l_3) \in S$ are of the form

$$\int\limits_{\Omega_\gamma} u = 0, \int\limits_{\Omega_\gamma} (u_{i,j} - u_{j,i}) = 0,\ i, j = 1, 2, 3,$$

i.e. $(u, v, w) \in U_1$. □

Now assume that the external forces satisfy the following condition

$$\int\limits_{\Omega_\gamma} g\rho = 0\ \ \forall \rho \in R(\Omega_\gamma). \tag{16}$$

In this case the problem (11)-(12) can be rewritten in the following form

$$(u, v, w) \in Q \cap U_1, \tag{17}$$

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u} + \rho - u) - \int\limits_{\Omega_\gamma} g(\bar{u} + \rho - u) + \int\limits_{\gamma} N(v)\varepsilon(\bar{v} + l_{12} - v) - \tag{18}$$

$$- \int\limits_{\gamma} M(w)\nabla(\bar{w} + l_3 - w) \geq 0$$

$$\forall (\bar{u}, \bar{v}, \bar{w}) \in Q \cap U_1,\ \forall (\rho, l_{12}, l_3) \in S,$$

or, by Proposition 2 below, in the following form

$$(u, v, w) \in Q \cap U_1, \tag{19}$$

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\tilde{u} - u) - \int\limits_{\Omega_\gamma} g(\tilde{u} - u) + \int\limits_{\gamma} N(v)\varepsilon(\tilde{v} - v) - \tag{20}$$

$$- \int\limits_{\gamma} M(w)\nabla\nabla(\tilde{w} - w) \geq 0\ \ \forall (\tilde{u}, \tilde{v}, \tilde{w}) \in Q.$$

To justify the equivalence of (17)-(18) and (19)-(20) given that (16) holds we need to have in our disposal the following statement.

Proposition 2. *For any* $(\tilde{u}, \tilde{v}, \tilde{w}) \in Q$ *there exist* $(\bar{u}, \bar{v}, \bar{w}) \in Q \cap U_1$ *and* $(\rho, l_{12}, l_3) \in S$ *such that*

$$(\tilde{u}, \tilde{v}, \tilde{w}) = (\bar{u}, \bar{v}, \bar{w}) + (\rho, l_{12}, l_3).$$

Proof. According to Proposition 1, we have

$$(\tilde{u}, \tilde{v}, \tilde{w}) = (\bar{u}, \bar{v}, \bar{w}) + (\rho, l_{12}, l_3);\ \ (\bar{u}, \bar{v}, \bar{w}) \in U_1,\ (\rho, l_{12}, l_3) \in S.$$

On the other hand,

$$[\tilde{u}_3] = [\bar{u}_3] + [\rho_3] \text{ on } \gamma.$$

Hence $[\bar{u}_3] \geq 0$ on γ, i.e. $(\bar{u}, \bar{v}, \bar{w}) \in Q \cap U_1$ what we needed. □

As a result, we arrive at the following assertion.

Theorem 1. *There exists a unique solution of the problem* (19)-(20) *provided that the condition* (16) *holds.*

The uniqueness of the solution can be easily checked by contradiction arguments. In so doing, we can take into account reasonings used to prove the coercivity of the functional E.

One more assertion takes place.

Theorem 2. *Problem formulations* (2)-(8) *and* (19)-(20) *are equivalent provided that the solutions are quite smooth.*

We do not provide a proof of this statement since in Section 3 we prove a similar assertion in a more difficult case.

Notice that the condition (16) is not only sufficient for the solvability of the problem (19)-(20) but it is necessary. Indeed, assume that the problem (19)-(20) has a solution. We can take $(\tilde{u}, \tilde{v}, \tilde{w}) = (u, v, w) \pm (\rho, l_{12}, l_3)$ as a test function in (20), where $(\rho, l_{12}, l_3) \in S$. This implies (16).

3 Rigid inclusion in Ω_γ

Let $\omega \subset \Omega$ be a small subdomain with a smooth boundary $\partial\omega$; we assume that $\partial\omega \cap \Gamma = \emptyset, (0, 0, 0) \in \omega$, see Fig.2. A unit normal vector to $\partial\omega$ is denoted by $s = (s_1, s_2, s_3)$. Denote $\Sigma = \omega \cap \gamma,\ \omega_\Sigma = \omega \setminus \bar{\Sigma}$. We assume that the geometry of the domain ω is such that Σ is a 2D domain with the Lipschitz boundary $\partial\Sigma$. In what follows, we suppose that the elasticity tensor depends on a positive parameter α inside the domain ω_Σ, and we aim to pass to the limit as $\alpha \to \infty$. Namely, let

$$A^\alpha = \begin{cases} A & \text{in} \quad \Omega_\gamma \setminus \bar{\omega} \\ \alpha A & \text{in} \quad \omega_\Sigma\ . \end{cases}$$

Like before, we assume that the condition (16) is fulfilled.

Denote $\sigma^\alpha(u) = A^\alpha \varepsilon(u)$. For any fixed $\alpha > 0$, we can find a unique solution of the problem like (19)-(20). Indeed, there exists a solution of the

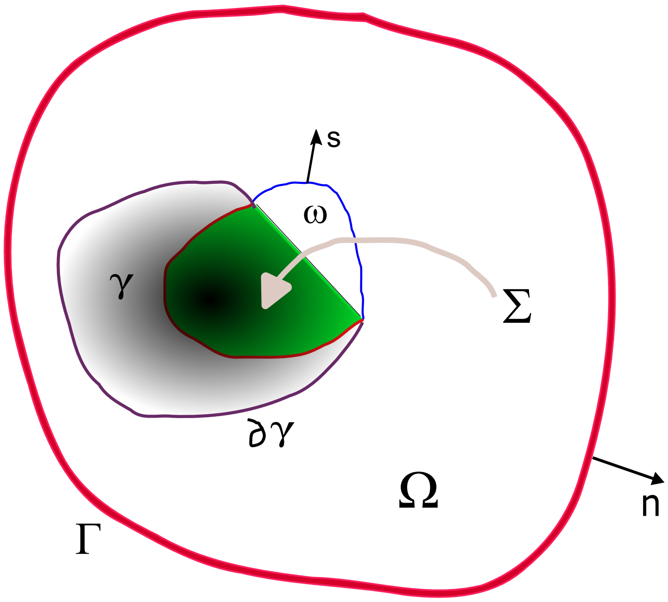

FIG. 2. Subdomain ω

problem

$$(u^\alpha, v^\alpha, w^\alpha) \in Q \cap U_1, \tag{21}$$

$$\int\limits_{\Omega_\gamma} \sigma^\alpha(u^\alpha)\varepsilon(\tilde{u} - u^\alpha) - \int\limits_{\Omega_\gamma} g(\tilde{u} - u^\alpha) + \int\limits_\gamma N(v^\alpha)\varepsilon(\tilde{v} - v^\alpha) - \tag{22}$$
$$- \int\limits_\gamma M(w^\alpha)\nabla\nabla(\tilde{w} - w^\alpha) \geq 0 \quad \forall (\tilde{u}, \tilde{v}, \tilde{w}) \in Q.$$

First, we obtain a priori estimates of the problem (21)-(22). From (21)-(22) it follows

$$\alpha \int\limits_{\omega_\Sigma} \sigma(u^\alpha)\varepsilon(u^\alpha) + \int\limits_{\Omega_\gamma \setminus \bar{\omega}} \sigma(u^\alpha)\varepsilon(u^\alpha) - \int\limits_{\Omega_\gamma} g u^\alpha + \int\limits_\gamma N(v^\alpha)\varepsilon(v^\alpha) - \tag{23}$$
$$- \int\limits_\gamma M(w^\alpha)\nabla\nabla w^\alpha = 0.$$

The arguments of Section 2 used to prove the coercivity of the functional E allow us to derive from (23) the estimate being uniform in $\alpha \geq \alpha_0 > 0$,

$$\|(u^\alpha, v^\alpha, w^\alpha)\|_U \leq c_1, \tag{24}$$

and moreover,

$$\int\limits_{\omega_\Sigma} \sigma(u^\alpha)\varepsilon(u^\alpha) \le \frac{c_2}{\alpha}. \tag{25}$$

In view of (24)-(25), we can assume that as $\alpha \to \infty$,

$$(u^\alpha, v^\alpha, w^\alpha) \to (u, v, w) \text{ weakly in } U; \;\; u|_{\omega_\Sigma} = \rho^0 \in R(\omega_\Sigma). \tag{26}$$

Introduce the space

$$U^\omega = \{(u, v, w) \in U \mid \;\; u|_{\omega_\Sigma} \in R(\omega_\Sigma)\}$$

and the set of admissible displacements suitable for the limit problem,

$$Q^\omega = \{(u, v, w) \in U^\omega \mid \;\; [u_3] \ge 0 \text{ on } \gamma;\, u = (u_1, u_2, u_3)\}.$$

Let us take $(\tilde{u}, \tilde{v}, \tilde{w}) \in Q^\omega$; in this case $(\tilde{u}, \tilde{v}, \tilde{w}) \in Q$. Hence, this element can be substituted in (22) as a test one. Passing to the limit as $\alpha \to \infty$, we obtain

$$(u, v, w) \in Q^\omega \cap U_1, \tag{27}$$

$$\int\limits_{\Omega_\gamma\setminus\bar{\omega}} \sigma(u)\varepsilon(\tilde{u} - u) - \int\limits_{\Omega_\gamma} g(\tilde{u} - u) + \int\limits_{\gamma\setminus\bar{\Sigma}} N(v)\varepsilon(\tilde{v} - v) - \tag{28}$$

$$- \int\limits_{\gamma\setminus\bar{\Sigma}} M(w)\nabla\nabla(\tilde{w} - w) \ge 0 \;\; \forall(\tilde{u}, \tilde{v}, \tilde{w}) \in Q^\omega.$$

As a result we prove the following assertion.

Theorem 3. *Solutions of the problem* (21)-(22) *converge in the sense* (26)*, as* $\alpha \to \infty$*, to the solution of the problem* (27)-(28) *provided that the condition* (16) *holds.*

Notice that we integrate over $\gamma \setminus \bar{\Sigma}$ (not over γ) in (28) since $(v, w) = \rho^0$ on Σ, and hence $N(v) = M(w) = 0$ on Σ.

We can provide the differential statement of the problem (27)-(28): it is necessary to find functions $u = (u_1, u_2, u_3)$ defined in Ω_γ, functions $v = (v_1, v_2), w$ defined on γ, as well as a function $\rho^0 = (\rho^0_1, \rho^0_2, \rho^0_3) \in R(\omega_\Sigma)$ such that

$$-\text{div}\sigma = g,\ \sigma = A\varepsilon(u) \text{ in } \Omega_\gamma \setminus \bar{\omega}, \tag{29}$$

$$-\text{div}N = [(\sigma_{13}, \sigma_{23})],\ \ N = B\varepsilon(v) \text{ on } \gamma \setminus \bar{\Sigma}, \tag{30}$$

$$-\nabla\nabla M = [\sigma_{33}],\ \ M = -D\nabla\nabla w \text{ on } \gamma \setminus \bar{\Sigma}, \tag{31}$$

$$\sigma n = 0 \ \text{ on } \Gamma;\ N\nu = 0,\ M^\nu = T^\nu = 0 \text{ on } \partial\gamma \setminus \partial\Sigma, \tag{32}$$

$$u_i^- = v_i,\ \ i = 1, 2;\ u_3^- = w;\ \ [u_3] \geq 0 \text{ on } \ \gamma, \tag{33}$$

$$\sigma_{33}^+ \leq 0,\ \sigma_{i3}^+ = 0,\ i = 1, 2;\ \ [u_3]\sigma_{33}^+ = 0 \text{ on } \gamma \setminus \bar{\Sigma}, \tag{34}$$

$$u = \rho^0 \text{ on } \omega_\Sigma;\ [u] = 0 \ \text{ on } \partial\omega, \tag{35}$$

$$\rho^0 = (v, w),\ \frac{\partial w}{\partial \nu} = \frac{\partial \rho_3^0}{\partial \nu} \text{ on } \partial\Sigma \setminus \partial\gamma, \tag{36}$$

$$-\int\limits_{\partial\omega} \sigma s \cdot \rho + \int\limits_{\partial\Sigma\setminus\partial\gamma} N\nu \cdot (\rho_1, \rho_2) + \int\limits_{\partial\Sigma\setminus\partial\gamma} M^\nu \frac{\partial \rho_3}{\partial \nu} - \tag{37}$$

$$- \int\limits_{\partial\Sigma\setminus\partial\gamma} T^\nu \cdot \rho_3 - \int\limits_{\omega_\Sigma} g\rho = 0 \ \ \forall \rho = (\rho_1, \rho_2, \rho_3) \in R(\omega_\Sigma),$$

$$\int\limits_{\Omega_\gamma} u = 0,\ \int\limits_{\Omega_\gamma} (u_{i,j} - u_{j,i}) = 0,\ \ i, j = 1, 2, 3. \tag{38}$$

The jump $u^+ - u^-$ of the function u with respect to the normal vector s on the surface $\partial\omega$ is denoted by $[u]$.

Notice that the functions v, w found from (27)-(28) are defined on γ. On the other hand, by the conditions $u|_{\omega_\Sigma} \in R(\omega_\Sigma)$, $(u_1^-, u_2^-, u_3^-) = (v, w)$ on γ, we formulate equilibrium equations (30)-(31) for v, w only on $\gamma \setminus \bar{\Sigma}$. These facts are included in (29)-(38), i.e in the differential statement of the problem (27)-(28).

The problem (29)-(38) describes an equilibrium state of the elastic body $\Omega_\gamma \setminus \bar{\omega}$ containing the thin elastic inclusion $\gamma \setminus \bar{\Sigma}$ and the volume rigid inclusion ω_Σ. Relations (36)-(37) can be seen as junction conditions. In particular, (37) implies that the principal vector of forces and the principal moment acting on the rigid inclusion ω_Σ are equal to zero.

The following assertion takes place.

Theorem 4. *Problem formulations (27)-(28) and (29)-(38) are equivalent provided that the solutions are quite smooth.*

Proof. Let (27)-(28) be fulfilled. First note that the equilibrium equation for the elastic body holds. Indeed, take in (28) the test functions of the form $(\tilde{u}, \tilde{v}, \tilde{w}) = (u, v, w) \pm (\varphi, 0, 0)$, $\ \varphi \in C_0^\infty(\Omega_\gamma \setminus \bar{\omega})^3$, assuming that φ is extended by zero outside of $\Omega_\gamma \setminus \bar{\omega}$. It implies

$$-\text{div}\sigma = g \ \text{ in } \ \Omega_\gamma \setminus \bar{\omega}$$

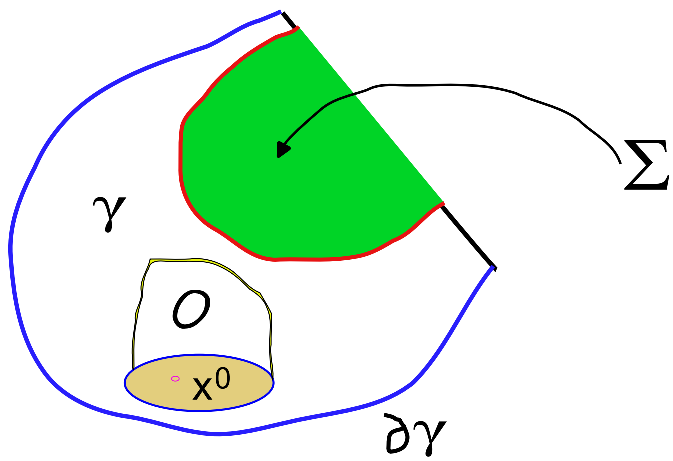

FIG. 3. Set $\bar{O} \cap \{x_3 \geq 0\}$

which is needed. Next, choose in (28) test functions $(\tilde{u}, \tilde{v}, \tilde{w}) = (u, v, w)\pm$ $\pm(\psi, 0, 0)$, where a support of the function $\psi = (\psi_1, \psi_2, \psi_3)$ is located in a small neighborhood of the boundary Γ. Integrating by parts in the relation obtained and taking into account the equilibrium equation, we derive

$$\int\limits_{\Gamma} \sigma n \cdot \psi = 0.$$

Since the functions ψ are arbitrary, this identity implies the first condition of (32).

To proceed, we prove that equilibrium equations from (30), (31) are fulfilled. To this end, the following test functions are taken in (28):

$$(\tilde{u}, \tilde{v}, \tilde{w}) = (u, v, w) \pm (\hat{u}, \hat{v}, \hat{w}), \;\; (\hat{u}, \hat{v}, \hat{w}) \in Q^{\omega}, \;\; [\hat{u}] = 0 \text{ on } \gamma.$$

This substitution gives

$$\int\limits_{\Omega_\gamma \setminus \bar{\omega}} (\sigma(u)\varepsilon(\hat{u}) - g\hat{u}) - \int\limits_{\omega_\Sigma} g\hat{u} + \int\limits_{\gamma \setminus \bar{\Sigma}} N(v)\varepsilon(\hat{v}) - \int\limits_{\gamma \setminus \bar{\Sigma}} M(w)\nabla\nabla\hat{w} = 0. \quad (39)$$

Denote by $m = (0, 0, 1)$ the unit normal vector to γ. Integrating by parts in (39) and taking into account the equilibrium equation from (29) and the first condition of (32), we obtain (with $\sigma = \sigma(u), N = N(v), M = M(w)$)

$$-\int\limits_{\gamma\setminus\bar\Sigma}[\sigma m]\hat u - \int\limits_{\omega_\Sigma} g\hat u - \int\limits_{\gamma\setminus\bar\Sigma}\mathrm{div}N\cdot\hat v - \int\limits_{\gamma\setminus\bar\Sigma}\nabla\nabla M\cdot\hat w - \tag{40}$$

$$-\int\limits_{\partial\omega}\sigma s\cdot\hat u + \int\limits_{\partial(\gamma\setminus\bar\Sigma)} N\nu\cdot\hat v + \int\limits_{\partial(\gamma\setminus\bar\Sigma)} M^\nu\hat w_\nu - \int\limits_{\partial(\gamma\setminus\bar\Sigma)} T^\nu\hat w = 0.$$

In so doing, the following Green's formula for a domain $G\subset\mathbb{R}^2$ with a smooth boundary ∂G is used: for smooth functions $w, M = \{M_{ij}\},\ i,j = 1,2$, we have [12]

$$-\int\limits_G M\cdot\nabla\nabla w = -\int\limits_G w\nabla\nabla M + \int\limits_{\partial G} M^q w_q - \int\limits_{\partial G} T^q w,$$

where q is outward unit vector to the boundary ∂G, $w_q = \frac{\partial w}{\partial q}$, and M^q, T^q are defined similar to (1). Temporarily assume that the test functions in (40) satisfy the conditions

$$\hat v = \hat w = \hat w_\nu = 0 \ \text{ on } \ \partial(\gamma\setminus\bar\Sigma).$$

In this case $\hat u \equiv 0$ in ω_Σ, hence, $\hat u = 0$ on $\partial\omega$. Moreover, notice that $[\sigma m] = [(\sigma_{13},\sigma_{23},\sigma_{33})]$ on $\gamma\setminus\bar\Sigma$. Then, in view of the boundary condition $\hat u = (\hat v, \hat w)$ on $\gamma\setminus\bar\Sigma$, the relation (40) implies

$$-\mathrm{div}N = [(\sigma_{13},\sigma_{23})],\ \ -\nabla\nabla M = [\sigma_{33}] \ \text{ on } \ \gamma\setminus\bar\Sigma,$$

i.e. the equilibrium equations for the thin inclusion are fulfilled, see (30), (31).

Going back to (40), by arbitrariness of $\hat v, \hat w, \hat w_\nu$ on $\partial\gamma\setminus\partial\Sigma$, we obtain the second group of boundary conditions (32). Thus, from (40) it follows

$$-\int\limits_{\partial\omega}\sigma s\cdot\hat u + \int\limits_{\partial\Sigma\setminus\partial\gamma} N\nu\cdot\hat v + \int\limits_{\partial\Sigma\setminus\partial\gamma} M^\nu\hat w_\nu - \int\limits_{\partial\Sigma\setminus\partial\gamma} T^\nu\hat w - \int\limits_{\omega_\Sigma} g\hat u = 0.$$

This identity coincides with (37), since $(\hat v,\hat w) = \rho,\ \hat w_\nu = \frac{\partial\rho_3}{\partial\nu}$ on $\partial\Sigma\setminus\partial\gamma$, $\rho\in R(\omega_\Sigma),\ \rho = (\rho_1,\rho_2,\rho_3)$.

Boundary conditions (34) can be derived from (27)-(28) by the following arguments. Take a test function in (28) in the form $(\tilde u,\tilde v,\tilde w) = (u,v,w) + (\bar u,\bar v,\bar w)$, where $\bar u = (\bar u_1,\bar u_2,\bar u_3)$, and the support of the function $\bar u$ is located in $\bar O\cap\{x_3\ge 0\}$; here O is a small neighbourhood of a point $x^0\in\gamma\setminus\bar\Sigma$, $\bar u_3\ge 0$ on $(\gamma\setminus\bar\Sigma)\cap\bar O$, see Fig. 3. This substitution provides the inequality

$$\int\limits_O \sigma(u)\varepsilon(\bar u) - \int\limits_O g\bar u \ge 0.$$

The first relations from (34) easily follow from this inequality.

Now, we prove the last relation from (34). Assume that $[u_3(x^0)] > 0,\ x^0\in\gamma\setminus\bar\Sigma$. Then it is possible to take in (28) a test function of the form $(\tilde u,\tilde v,\tilde w) =$

$(u,v,w) \pm \beta(\varphi,0,0)$, where supp $\varphi \subset \bar{O} \cap \{x_3 \geq 0\}$, $\beta \in \mathbb{R}$, β is small, see Fig.3. This implies

$$\int\limits_{O} \sigma(u)\varepsilon(\varphi) - \int\limits_{O} g\varphi = 0.$$

Hence, integrating by parts, we obtain $\sigma_{33}^{+}(x^0) = 0$. On the other hand, assuming that $\sigma_{33}^{+}(x^0) < 0$, the equality $[u_3(x^0)] = 0$ easily follows. Consequently, the last equality of (34) is derived.

Equalities (35), (36) are consequences of the relation $(u,v,w) \in Q^{\omega}$.

Thus, all relations (29)-(38) are derived from (27)-(28).

Let us prove the converse. Assume that (29)-(38) hold. Multiply the first equations of (29)-(31) by $\tilde{u}-u$, $\tilde{v}-v$, $\tilde{w}-w$, where $(\tilde{u},\tilde{v},\tilde{w}) \in Q^{\omega}$, and integrate over $\Omega_\gamma \setminus \bar{\omega}$, $\gamma \setminus \bar{\Sigma}$, $\gamma \setminus \bar{\Sigma}$, respectively. We obtain

$$\int\limits_{\Omega_\gamma\setminus\bar{\omega}} (\mathrm{div}\sigma + g)(\tilde{u}-u) + \int\limits_{\gamma\setminus\bar{\Sigma}} \mathrm{div}N(\tilde{v}-v) + \int\limits_{\gamma\setminus\bar{\Sigma}} \nabla\nabla M(\tilde{w}-w) +$$
$$+ \int\limits_{\gamma\setminus\bar{\Sigma}} [(\sigma_{13},\sigma_{23})](\tilde{v}-v) + \int\limits_{\gamma\setminus\bar{\Sigma}} [\sigma_{33}](\tilde{w}-w) = 0.$$

From here, it follows

$$\int\limits_{\Omega_\gamma\setminus\bar{\omega}} \sigma(u)\varepsilon(\tilde{u}-u) - \int\limits_{\Omega_\gamma} g(\tilde{u}-u) + \int\limits_{\gamma\setminus\bar{\Sigma}} N(v)\varepsilon(\tilde{v}-v) -$$
$$- \int\limits_{\gamma\setminus\bar{\Sigma}} M(w)\nabla\nabla(\tilde{w}-w) + \int\limits_{\gamma\setminus\bar{\Sigma}} [\sigma m(\tilde{u}-u)] - \int\limits_{\gamma\setminus\bar{\Sigma}} [\sigma m](\tilde{u}^- - u^-) +$$
$$= \int\limits_{\partial\omega} \sigma s(\tilde{u}-u) + \int\limits_{\omega_\Sigma} g(\tilde{u}-u) - \int\limits_{\partial(\gamma\setminus\bar{\Sigma})} N\nu(\tilde{v}-v) -$$
$$- \int\limits_{\partial(\gamma\setminus\bar{\Sigma})} M^{\nu}(\tilde{w}_\nu - w_\nu) + \int\limits_{\partial(\gamma\setminus\bar{\Sigma})} T^{\nu}(\tilde{w}-w) = 0. \quad (41)$$

By the second boundary conditions (32) and the identity (37), the last five terms in the left-hand side of (41) cancel since $(v,w) = \rho^0$, $(\tilde{v},\tilde{w}) = \rho$, $\tilde{w}_\nu = \frac{\partial\rho_3}{\partial\nu}$, $w_\nu = \frac{\partial\rho_3^0}{\partial\nu}$ on $\partial\Sigma \setminus \partial\gamma$, $\rho = (\rho_1,\rho_2,\rho_3) \in R(\omega_\Sigma)$. Consequently, the relation (41) implies (28) if the following inequality holds

$$\int\limits_{\gamma\setminus\bar{\Sigma}} [\sigma m(\tilde{u}-u)] - \int\limits_{\gamma\setminus\bar{\Sigma}} [\sigma m](\tilde{u}^- - u^-) \leq 0. \quad (42)$$

On the other hand, the inequality (42) is valid in view of (33)-(34). Thus, we have proved that (29)-(38) imply (27)-(28).

□

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