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WEAK SOLUTIONS OF THE NAVIER–STOKES EQUATIONS WITH A SHORT-TERM INTENSE INITIAL PULSE

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Abstract: We address the system of Navier–Stokes equations that models a homogeneous viscous incompressible fluid in the presence of a short-term intense pulse starting at an initial moment in time. The existence of a weak Leray–Hopf solution to the initial-boundary value problem for this system in the case when the pulse duration is fixed is guaranteed by the well-known theory. On a rigorous mathematical level, we carry out the limiting transition in the problem as the pulse duration tends to zero, while the cumulative impact of the pulse remains constant. As a result, we prove that the family of weak Leray–Hopf solutions to the problem under consideration has a subsequence converging to a weak solution of the initial-boundary value problem for the system of classical Navier–Stokes equations supplemented by the "corrected" initial velocity field inheriting complete information about the profile and cumulative impact of the original pulse. The "corrected" initial velocity field is found out as the solution of an additional limit system of equations of inviscid fluid derived at the microscopic

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("fast") timescale, which is the characteristic timescale of the pulse duration. At the end of the article, we identify two particular cases in which this system can be solved explicitly, which leads to an explicit algebraic expression of the "corrected" initial velocity.

Keywords: viscous incompressible fluid, boundary value problem, weak solution, impulsive partial differential equation, infinitesimal initial layer.

1 Introduction

We study the initial-boundary value problem for the system of Navier–Stokes equations of dynamics of homogeneous viscous incompressible fluid in the presence of a short-term pulse right after an initial moment in time:

$$\partial_t \boldsymbol{v}_n + \mathrm{div}_x(\boldsymbol{v}_n \otimes \boldsymbol{v}_n) = \mu \Delta_x \boldsymbol{v}_n + \varphi_n(t)(\boldsymbol{b}(\boldsymbol{x}) + \mathbb{B}(\boldsymbol{x})\boldsymbol{v}_n) - \nabla_x p_{*n} + \boldsymbol{f} \quad \text{in } Q_T, \tag{1a}$$

$$\mathrm{div}_x \boldsymbol{v}_n = 0 \quad \text{in } Q_T, \tag{1b}$$

$$\boldsymbol{v}_n(\cdot, 0) = \boldsymbol{v}_0 \quad \text{in } \Omega, \tag{1c}$$

$$\boldsymbol{v}_n = 0 \quad \text{on } \partial\Omega \times (0,T). \tag{1d}$$

In (1) and further, $\Omega \subset \mathbb{R}^d_x$ is a bounded domain with a smooth boundary $\partial\Omega \in C^3$, $d \in \{2,3\}$ is the dimension of the space of physical positions $\boldsymbol{x}$, t is the time variable, $T = \mathrm{const} > 0$ is an arbitrarily given fixed moment of time, and $Q_T = \Omega \times (0,T)$ is the space-time cylinder.

Velocity field $\boldsymbol{v}_n = \boldsymbol{v}_n(\boldsymbol{x}, t)$ and pressure distribution $p_{*n} = p_{*n}(\boldsymbol{x}, t)$ are the sought functions.

A given positive viscosity coefficient μ is constant. Functions $\boldsymbol{b}$: $\Omega \mapsto \mathbb{R}^d$, $\mathbb{B}$: $\Omega \mapsto \mathbb{R}^{d\times d}$, and φ_n: $[0,T] \mapsto \mathbb{R}_+$ are also given and characterize the impulsive effect. They are the respective vector- and matrix-valued functions of intensity of the pulse and a temporal profile of the pulse. We assume that these functions meet the following regularity and structural demands:

the components of matrix $\mathbb{B} = (b_{ij})$ and vector $\boldsymbol{b} = (b_i)$

are measurable and bounded in Ω:

$$b_i,\, b_{ij} \in L^\infty(\Omega), \quad |b_i(\boldsymbol{x})|,\, |b_{ij}(\boldsymbol{x})| \le M_0 \text{ a.e. in } \Omega, \quad i,j = 1,\dots,d, \tag{2}$$

with some positive constant M_0,

function $\varphi_n = \varphi_n(t)$ approximates the Dirac delta function $\delta_{(t=0)}$, more precisely, for each natural n, $n \ge n_0 = \left[\dfrac{1}{T}\right] + 2$, it has the form

$$\varphi_n(t) = n\Phi(nt), \quad t \in [0,T], \tag{3}$$

where $\Phi = \Phi(\vartheta)$ is a nonnegative smooth function supported on the segment $\{0 \le \vartheta \le 1\}$ with the mean value equal to unity:

$$\Phi \ge 0 \ \text{ in } \mathbb{R}, \quad \Phi \in C_0^1(\mathbb{R}), \quad \operatorname{supp}\Phi \subset [0,1], \quad \int_0^1 \Phi(\vartheta)\, d\vartheta = 1. \tag{4}$$

Note that

$$\int_0^1 \varphi_n(t)\psi(t)dt \underset{n\to+\infty}{\longrightarrow} \psi(0), \quad \forall\, \psi \in C[0,1],$$

which means that $\varphi_n \underset{n\to+\infty}{\longrightarrow} \delta_{(t=0)}$ weakly* in the space of Radon measures on $[0,1]$. Also note that

$$\int_0^t \varphi_n(s)ds \le 1, \ \forall\, t \in [0,T], \quad \int_0^T \varphi_n(s)ds = 1 \tag{5}$$

due to (3) and (4).

Vector-function $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x},t)$ is a given distributed mass force satisfying the regularity demand

$$\boldsymbol{f} \in C([0,T]; W^{-1,2}(\Omega)), \quad \operatorname{div}_x \boldsymbol{f} = 0 \text{ (in the sense of distributions)}, \tag{6}$$

vector-function $\boldsymbol{v}_0 = \boldsymbol{v}_0(\boldsymbol{x})$ is a given initial velocity field such that

$$\boldsymbol{v}_0 \in L^2(\Omega)^d, \quad \operatorname{div}_x \boldsymbol{v}_0 = 0 \quad \text{ in } \Omega, \tag{7}$$

and the constant density of fluid is scaled out to unity.

The peculiarity of the problem under consideration obviously lies in the consideration of the pulse action. We can note that the results available in the literature on the Navier–Stokes equations with pulses are currently very limited. In this regard, it is worth mentioning the article [7] devoted to the study of the system of Navier–Stokes equations of an incompressible fluid with multiple instantaneous pulses occurring at various moments in time and collectively providing the system with the dissipative property, and the article [14], in which the system of Navier–Stokes equations of a compressible fluid is supplemented with initial data for density and velocity, depending on a small parameter characterizing an initial instantaneous pulse, and the limiting transition is considered as this small parameter tends to zero.

As for the present paper, the presence of the impulsive term

$$\varphi_n(t)(\boldsymbol{b}(\boldsymbol{x}) + \mathbb{B}(\boldsymbol{x})\boldsymbol{v}_n)$$

from a mathematical point of view attracts significant interest, which consists in the following.

Firstly, if we formally substitute $(\boldsymbol{v}_n, p_{*n})$ by $(\boldsymbol{v}, p_*)$ and the impulsive term by the expression $\delta_{(t=0)}(\boldsymbol{b}(\boldsymbol{x}) + \mathbb{B}(\boldsymbol{x})\boldsymbol{v})$ in (1a), then from (1a), (1c) we deduce the subsystem, which is in the sense of distributions equivalent to the subsystem consisting of the momentum equation

$$\partial_t \boldsymbol{v} + \operatorname{div}_x\big(\boldsymbol{v} \otimes \boldsymbol{v}\big) = \mu\Delta_x \boldsymbol{v} - \nabla_x p_* + \boldsymbol{f} \tag{8}$$

and the "corrected" initial condition

$$\boldsymbol{v}(\boldsymbol{x}, 0+) = \boldsymbol{v}_0(\boldsymbol{x}) + \mathcal{P}_{\mathrm{sol}}(\boldsymbol{b} + \mathbb{B}\boldsymbol{v}_0)(\boldsymbol{x}), \tag{9}$$

where $\mathcal{P}_{\rm sol}$ is the operator of projection onto the subspace of finite solenoidal vector-functions in Ω, for any sufficiently regular $\boldsymbol{\xi}$ acting by the rule

$$\mathcal{P}_{\rm sol}(\boldsymbol{\xi}) = \boldsymbol{\xi} + \nabla_x \overline{\pi}_*,$$

where $\Delta_x \overline{\pi}_* = -\,{\rm div}_x \boldsymbol{\xi}$. (Obviously, the form of subsystem (1b), (1d) remains intact.) At the same time, the numerous observations in the theory of impulsive ordinary and partial differential equations signal that, in cases when $\mathbb{B} \neq 0$, subsystem (8)–(9) most likely may not be the correct limit form of (1a), (1c) as $n \to +\infty$, see, for example [10, 13]. This means that a thorough and mathematically rigorous analysis of the limiting passage as $n \to +\infty$ is needed to establish the correct limit mode when modeling a short-term pulse by an instantaneous one.

Secondly, there already exists a fairly extensive theory of impulsive abstract, ordinary and partial differential equations, involving non-instantaneous impulsive terms of type $\varphi_n(t)(\boldsymbol{b}(\boldsymbol{x}) + \mathbb{B}(\boldsymbol{x})\boldsymbol{v}_n)$, and this theory is gradually evolving. Within the framework of this theory, the study of abstract evolutionary equations is well represented in monographs [1, 30], the study of ordinary differential equations began, as far as we know, with article [15], and a significant part of the further analysis of impulsive ordinary differential equations, also called "generalized differential equations", is described in detail in monographs [11, 23]. The research presented in the present article belongs to the field of impulsive partial differential equations. It complements our cycle of work on impulsive parabolic [5, 17] and pseudo-parabolic [4, 16, 18] equations and on the system of the Kelvin–Voigt equations with a pulse [3]. More specifically, the present study can be considered as a continuation of the work [3] in connection with the following observation. In the system of the Kelvin–Voigt equations with a pulse, the momentum equation

$$\begin{aligned}&\partial_t \boldsymbol{v}_n + {\rm div}_x(\boldsymbol{v}_n \otimes \boldsymbol{v}_n) = \\ &\quad = \mu \Delta_x \boldsymbol{v}_n + \kappa \Delta_x \partial_t \boldsymbol{v}_n + \varphi_n(t)(\boldsymbol{b}(\boldsymbol{x}) + \mathbb{B}(\boldsymbol{x})\boldsymbol{v}_n) - \nabla_x p_{*n} + \boldsymbol{f} \quad \text{in } Q_T\end{aligned}$$

can be interpreted as a regularization of the momentum equation (1a), since the presence of the damping term $\kappa \Delta_x \partial_t \boldsymbol{v}_n$, with $\kappa > 0$, enables the higher regularity of the weak solution and allows for the additional uniform (in n, not in κ) estimates for the family of weak solutions, see [3, Propos. 1, 2]. In comparison with [3], due to the absence of the term $\kappa \Delta_x \partial_t \boldsymbol{v}_n$, for weak solutions of the Navier–Stokes system (1) we are more limited in regularity properties. This circumstance forces us to apply in this article a more subtle technique compared to [3] in order to perform and strictly justify the transition to the limit in (1) as $n \to +\infty$.

From a physical viewpoint, the terms of type $\varphi_n(t)(\boldsymbol{b}(\boldsymbol{x}) + \mathbb{B}(\boldsymbol{x})\boldsymbol{v}_n)$ represent a short but very intensive force action, which can be called a *non-instantaneous impulsive action*. In the simplest case, when $\boldsymbol{b} \neq 0$ and $\mathbb{B} = 0$, we can note that the study conducted in the present article is in good agreement with the classical formulations of problems on exterior instantaneous force action on hydrodynamic flows [6, 20, 22, 28]. More certainly, passing to the limit in (1) as $n \to +\infty$, in this case we derive the "corrected" initial condition (9) with $\mathbb{B} = 0$, which exactly corresponds to the impulsive source term $\boldsymbol{F} = \delta_{(t=0)}\boldsymbol{b}(\boldsymbol{x})$ considered in [6, 20, 22, 28], for details see Theorem 4 and Remark 3 in the end of the article.

More sophisticated cases, when $\mathbb{B} \neq 0$ and, therefore, the non-instantaneous pulse depends on the solution, also have significant physical meaning. For example, such a case arises in optimal control theory: following [12, Ch. 7, § 1.2] one can rewrite (1a) in the form of the system of the momentum equation and the linear algebraic equation:

$$\partial_t \boldsymbol{v}_n + \operatorname{div}_x(\boldsymbol{v}_n \otimes \boldsymbol{v}_n) = \mu \Delta_x \boldsymbol{v}_n - \nabla_x p_{*n} + \boldsymbol{f} + \boldsymbol{F}_n, \tag{10a}$$

$$\boldsymbol{F}_n = \varphi_n(t)(\boldsymbol{b} + \mathbb{B}\boldsymbol{v}_n), \tag{10b}$$

which models a feedback control system, where the vector-function $\boldsymbol{F}_n$ of control depends on the state variable — the velocity $\boldsymbol{v}_n$ — through a direct control mechanism, expressed by the linear algebraic equation (10b). In terms of the optimal control theory, the mapping $(\boldsymbol{x}, t, \boldsymbol{v}) \mapsto \varphi_n(t)\big(\boldsymbol{b}(\boldsymbol{x}) + \mathbb{B}(\boldsymbol{x})\boldsymbol{v}\big)$ is the characteristic function of the control mechanism and the state variable $\boldsymbol{v}$ plays the role of a feedback signal [21, Ch. 1, § 2]. Obviously, the presence of function φ_n in system (10) and, consequently, in equation (1a) reflects the circumstance that the feedback control is very intensive and very shortly distributed in time after the initial moment $t = 0$.

Another interesting example of the case of $\mathbb{B} \neq 0$ concerns fluid flows, in which impulsive actions can be caused not only by external, but also by internal (i.e., in-flow) processes. In this regard, the Navier–Stokes equations of the form (1a)–(1b) can naturally be used for mathematical modeling of the so-called *active fluids*. An active fluid is a fluid containing a great number of microscopic self-propelling active particles (agents) that convert their internal energy or energy from the environment into mechanical work and thereby collectively affect the flow of the fluid on a macroscopic scale. Active fluids are found both in living matter, for example, in the form of cellular fluids or dense bacterial suspensions (see [2, 9] and references therein), and in artificial systems, for example, in the form of polar gels [25, 29]. The appearance of the macroscopic impulsive term $\varphi_n(t)(\boldsymbol{b} + \mathbb{B}\boldsymbol{v}_n)$ in (1a), based on the effects produced by microscopic active agents in the active fluid at the microscale, can be loosely explained as follows.

Assume initially that there are N self-propelling agents, each of which produces the respective drag force

$$\boldsymbol{F}_i(t) = |\Omega_i|\varphi_n(t)\big(\boldsymbol{b}(\boldsymbol{x}_i(t)) + \mathbb{B}(\boldsymbol{x}_i(t))\boldsymbol{v}_n(\boldsymbol{x}_i(t), t)\big), \quad i = 1, \ldots, N,$$

where $\boldsymbol{x}_i(t)$ is the trajectory of the i-th agent such that $\boldsymbol{x}_i(t) \in \Omega_i$ and $\{\Omega_i\}_{i=1,\ldots,N}$ is a partition of the flow domain Ω: $\displaystyle\bigcup_{i=1}^{N} \overline{\Omega}_i = \overline{\Omega}$, $\Omega_i \cap \Omega_j = \emptyset$

for $i \neq j$. The aggregate drag force induced by these N agents is

$$\boldsymbol{F}_N = \sum_{i=1}^{N} \delta_{(x=x_i(t))} \boldsymbol{F}_i(t).$$

Now, letting the number of active agents tend to infinity and simultaneously assuming that $\lim\limits_{N\to+\infty} \sup\limits_{i=1,\dots,N} |\Omega_i| = 0$, we deduce that the sequence of drag forces $\boldsymbol{F}_N$ converges to the limit drag force

$$\boldsymbol{F}_n(\boldsymbol{x},t) = \varphi_n(t)\big(\boldsymbol{b}(\boldsymbol{x}) + \mathbb{B}(\boldsymbol{x})\boldsymbol{v}_n(\boldsymbol{x},t)\big)$$

in the weak* sense, as $N \to +\infty$, at least formally. Thus, by choosing $\boldsymbol{b}$ and $\mathbb{B}$ appropriately, we can simulate various types of macroscopic impulsive drag forces induced by translational and rotational impulsive movements of ensembles of self-propelling agents. These agents are, in fact, 'active-passive' agents, since they are active for a very short period of time to collectively generate a macroscopic pulse.

Now, let us briefly describe the further organization of the article. In relation with fixed values of parameter n, in Section 2, we revisit the well-known theory of existence and uniqueness of weak solutions to Navier–Stokes equations. In Section 3, we provide the rigorous formulation of the main results, which are the results of passing to the limit as $n \to +\infty$. Section 4 is devoted to construction of the uniform (in n) estimates of solutions to problem (1). In Section 5, we prove the main results of the article. Finally, in Section 6, for the limit model, we make a few remarks on uniqueness and regularity of solutions and discuss two simple examples of explicit "corrected" initial velocity fields.

2 Well-posedness of problem (1) for fixed n

The solution of problem (1) is understood in the weak sense. In order to formulate a definition of weak solution and for the further considerations, we introduce the following well-known functional spaces:

$$\begin{aligned}
\boldsymbol{\mathcal{V}} &:= \big\{\boldsymbol{v} \in C_0^\infty(\Omega)^d :\ \mathrm{div}_x \boldsymbol{v} = 0\big\},\\
\boldsymbol{H} &:= \text{the closure of } \boldsymbol{\mathcal{V}} \text{ in the norm of } L^2(\Omega)^d,\\
\boldsymbol{V}^l &:= \text{the closure of } \boldsymbol{\mathcal{V}} \text{ in the norm of } W_0^{l,2}(\Omega)^d, \quad l \in \mathbb{N},\\
\boldsymbol{V}^{-l} &:= \text{the dual space of } \boldsymbol{V}^l, \quad l \in \mathbb{N}.
\end{aligned}$$

In the case $l = 1$, we denote $\boldsymbol{V}^1$ simply by $\boldsymbol{V}$. Note that conditions (6) and (7) can be rewritten in the respective equivalent forms $\boldsymbol{f} \in C([0,T];\boldsymbol{V}^{-1})$ and $\boldsymbol{v}_0 \in \boldsymbol{H}$.

Now, for each $n \geq n_0$ ($n \in \mathbb{N}$), we introduce the notion of a weak Leray–Hopf solution to problem (1) similarly to the classical works, see, for example, in [8, Ch. V, Sec. 1], [27, Ch. 3].

Definition 1. *Vector-function* $\boldsymbol{v}_n = \boldsymbol{v}_n(\boldsymbol{x}, t)$ *is called* a weak Leray–Hopf solution *to problem* (1) *if it has the following properties:*

(i) *the regularity requirements*

$$\begin{aligned} &\boldsymbol{v}_n \in L^{\infty}(0,T;\boldsymbol{H}) \cap L^2(0,T;\boldsymbol{V}), \quad \partial_t \boldsymbol{v}_n \in L^{4/d}(0,T;\boldsymbol{V}^{-1}), \\ &\boldsymbol{v}_n \in C([0,T];\boldsymbol{H}) \ \text{if } d=2, \quad \boldsymbol{v}_n \in C_{\text{weak}}([0,T];\boldsymbol{H}) \ \text{if } d=3 \end{aligned} \tag{11}$$

hold true;

(ii) *the integral equality*

$$\begin{aligned} &\int_0^T \big\langle \partial_t \boldsymbol{v}_n(\cdot,t), \boldsymbol{w}(\cdot,t)\big\rangle_{\boldsymbol{V}^{-1},\boldsymbol{V}} dt \\ &+ \int_{Q_T} \big(\operatorname{div}_x(\boldsymbol{v}_n \otimes \boldsymbol{v}_n)\cdot \boldsymbol{w} + \mu \nabla_x \boldsymbol{v}_n : \nabla_x \boldsymbol{w}\big) d\boldsymbol{x}dt = \\ &= \int_0^T \varphi_n(t) \int_\Omega (\boldsymbol{b}(\boldsymbol{x}) + \mathbb{B}(\boldsymbol{x})\boldsymbol{v}_n)\cdot \boldsymbol{w}\, d\boldsymbol{x}dt + \int_0^T \big\langle \boldsymbol{f}(\cdot,t), \boldsymbol{w}(\cdot,t)\big\rangle_{\boldsymbol{V}^{-1},\boldsymbol{V}} dt \end{aligned} \tag{12}$$

is valid for all test vector-functions $\boldsymbol{w} \in L^{4/(4-d)}(0,T;\boldsymbol{V})$*;*

(iii) *in the case* $d=2$*, the initial condition* (1c) *holds in the sense of the strong trace in* $\boldsymbol{H}$*, i.e.*

$$\|\boldsymbol{v}_n(\cdot,t) - \boldsymbol{v}_0\|_{\boldsymbol{H}} \underset{t\to 0+}{\longrightarrow} 0; \tag{13a}$$

in the case $d=3$*, the initial condition* (1c) *holds in the sense of the weak trace in* $\boldsymbol{H}$*, i.e.*

$$\int_\Omega \boldsymbol{v}_n(\boldsymbol{x},t)\cdot \boldsymbol{w}(\boldsymbol{x})d\boldsymbol{x} \underset{t\to 0+}{\longrightarrow} \int_\Omega \boldsymbol{v}_0(\boldsymbol{x})\cdot \boldsymbol{w}(\boldsymbol{x})d\boldsymbol{x}, \quad \forall\, \boldsymbol{w} \in \boldsymbol{H}; \tag{13b}$$

(iv) *the energy inequality*

$$\begin{aligned} &\frac{1}{2}\|\boldsymbol{v}_n(\cdot,t)\|^2_{\boldsymbol{H}} + \mu \int_0^t \int_\Omega |\nabla_x \boldsymbol{v}_n|^2 d\boldsymbol{x}dt' \\ &\le \frac{1}{2}\|\boldsymbol{v}_0\|^2_{\boldsymbol{H}} + \int_0^t \varphi_n(t') \int_\Omega (\boldsymbol{b}(\boldsymbol{x}) + \mathbb{B}(\boldsymbol{x})\boldsymbol{v}_n)\cdot \boldsymbol{v}_n d\boldsymbol{x}dt' \\ &\qquad\qquad + \int_0^t \big\langle \boldsymbol{f}(\cdot,t'), \boldsymbol{v}_n(\cdot,t')\big\rangle_{\boldsymbol{V}^{-1},\boldsymbol{V}} dt' \end{aligned} \tag{14}$$

holds for all $t \in (0,T]$.

In (11) by $C_{\text{weak}}([0,T];\boldsymbol{H})$ we denote the subspace of $L^\infty(0,T;\boldsymbol{H})$ consisting of all weakly continuous mappings from $[0,T]$ into $\boldsymbol{H}$: each function $\boldsymbol{\phi} \in C_{\text{weak}}([0,T];\boldsymbol{H})$ satisfies the limiting relation

$$\int_\Omega \boldsymbol{\phi}(\boldsymbol{x},t_1)\cdot \boldsymbol{\psi}(\boldsymbol{x})d\boldsymbol{x} \underset{t_1\to t_0}{\longrightarrow} \int_\Omega \boldsymbol{\phi}(\boldsymbol{x},t_0)\cdot \boldsymbol{\psi}(\boldsymbol{x})d\boldsymbol{x}, \ \ \forall\, \boldsymbol{\psi} \in \boldsymbol{H}, \ \ \forall\, t_0, t_1 \in [0,T].$$

In (12), (14) and further, by $\langle \cdot,\cdot\rangle_{\boldsymbol{X}^*,\boldsymbol{X}}$ we standardly denote the duality bracket between a Banach space $\boldsymbol{X}$ and its dual space of functionals $\boldsymbol{X}^*$.

In the case $\boldsymbol{b} = 0$ and $\mathbb{B} = 0$, system (1) is the well-known initial-boundary value problem for classical Navier–Stokes equations, for which an extensive theory of well-posedness has been built over the past century. A natural modification of the proofs from [27, Ch. 3, Sec. 3] leads to the following results for problem (1) in the cases when $\boldsymbol{b} \neq 0$ and $\mathbb{B} \neq 0$.

Proposition 1. **1.** *Assume conditions (2)–(7) hold and $d \in \{2,3\}$. Then, for every fixed $n \geq n_0$ ($n \in \mathbb{N}$), problem (1) has at least one weak Leray–Hopf solution $\boldsymbol{v}_n$ in the sense of Definition 1.*

2. *Let $d = 2$. Then, under conditions (2)–(7), a weak Leray–Hopf solution to problem (1) is unique for every fixed $n \geq n_0$ ($n \in \mathbb{N}$).*

Additionally, we construct a set of uniform in n estimates for the family of weak Leray–Hopf solutions, as follows.

Proposition 2. *The family $\{\boldsymbol{v}_n\}_{n=1,2,\dots}$ of weak Leray–Hopf solutions to problem (1) satisfies the bounds*

$$\|\boldsymbol{v}_n\|_{L^\infty(0,T;\boldsymbol{H})} + \|\boldsymbol{v}_n\|_{L^2(0,T;\boldsymbol{V})} \leq C_0, \tag{15}$$

$$\|\partial_t \boldsymbol{v}_n\|_{L^1(0,T;\boldsymbol{V}^{-3})} \leq C_0, \tag{16}$$

with a constant C_0 independent of n.

We give a proof to Proposition 2 further in Section 4.

3 Formulation of the main results

The main results of this article deal with the passage to the limit in problem (1), as $n \to +\infty$. This limiting passage is based on the uniform in n estimates (15) and (16).

Theorem 1. *Assume conditions (2)–(7) hold and $\{\boldsymbol{v}_n\}_{n\geq n_0}$ is the family of weak Leray–Hopf solutions to problem (1) in the sense of Definition 1. Then the following assertions hold true.*

1. *There exist a subsequence of $\{\boldsymbol{v}_n\}_{n\geq n_0}$, still labeled by n, and a limit vector-function $\boldsymbol{v} \in L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{H})$ such that*

$$\begin{aligned} \boldsymbol{v}_n \underset{n\to+\infty}{\longrightarrow} \boldsymbol{v} &\text{ strongly in } L^2(0,T;\boldsymbol{H}), \text{ weakly in } L^2(0,T;\boldsymbol{V}), \\ &\text{weakly}^\star \text{ in } L^\infty(0,T;\boldsymbol{H}). \end{aligned} \tag{17}$$

In other words, $\{\boldsymbol{v}_n\}_{n\geq n_0}$ is relatively compact in $L^2(0,T;\boldsymbol{H})$, relatively weakly compact in $L^2(0,T;\boldsymbol{V})$, and relatively weakly* compact in $L^\infty(0,T;\boldsymbol{H})$.

2. *The family of rescaled solutions $\{\overline{\boldsymbol{v}}_n\}_{n\geq n_0}$, $\overline{\boldsymbol{v}}_n\colon \Omega \times [0,1] \mapsto \mathbb{R}^d$, defined by the formula*

$$\overline{\boldsymbol{v}}_n(\boldsymbol{x},\vartheta) \overset{\text{def}}{=} \boldsymbol{v}_n\Big(\boldsymbol{x}, \frac{\vartheta}{n}\Big), \quad \vartheta \in [0,1], \tag{18}$$

is relatively weakly* compact in $L^\infty(0,1;\boldsymbol{H})$: there exist a subsequence of $\{\overline{\boldsymbol{v}}_n\}_{n\geq n_0}$, still labeled by n, and a limit vector-function

$\overline{\boldsymbol{v}} \in L^\infty(0,1;\boldsymbol{H})$ *such that*

$$\overline{\boldsymbol{v}}_n \underset{n\to+\infty}{\longrightarrow} \overline{\boldsymbol{v}} \quad \textit{weakly}^\star \textit{ in } L^\infty(0,1;\boldsymbol{H}). \tag{19}$$

3. *The limit vector-function* $\overline{\boldsymbol{v}} = \overline{\boldsymbol{v}}(\boldsymbol{x},\vartheta)$ *is the* unique *strong solution to the initial-boundary value problem for the system of equations of an inviscid fluid:*

$$\partial_\vartheta \overline{\boldsymbol{v}} = \Phi(\vartheta)\big(\boldsymbol{b}(\boldsymbol{x}) + \mathbb{B}(\boldsymbol{x})\overline{\boldsymbol{v}}\big) - \nabla_x \overline{p}_* \quad \textit{in } \Omega\times(0,1), \tag{20a}$$

$$\operatorname{div}_x \overline{\boldsymbol{v}} = 0 \quad \textit{in } \Omega\times(0,1), \tag{20b}$$

$$\overline{\boldsymbol{v}}(\cdot,0) = \boldsymbol{v}_0 \quad \textit{in } \Omega, \tag{20c}$$

$$\overline{\boldsymbol{v}} = 0 \quad \textit{on } \partial\Omega\times(0,1). \tag{20d}$$

4. *The limit vector-function* $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x},t)$ *is a weak solution to the initial boundary value problem for the system of classical Navier–Stokes equations:*

$$\partial_t \boldsymbol{v} + \operatorname{div}_x\big(\boldsymbol{v}\otimes\boldsymbol{v}\big) = \mu\Delta_x \boldsymbol{v} - \nabla_x p_* + \boldsymbol{f} \quad \textit{in } Q_T, \tag{21a}$$

$$\operatorname{div}_x \boldsymbol{v} = 0 \quad \textit{in } Q_T, \tag{21b}$$

$$\boldsymbol{v}(\cdot,0+) = \overline{\boldsymbol{v}}(\cdot,1) \quad \textit{in } \Omega, \tag{21c}$$

$$\boldsymbol{v} = 0 \quad \textit{on } \partial\Omega\times(0,T). \tag{21d}$$

In (21c)*, the initial vector-function* $\boldsymbol{v}(\cdot,0+)$ *is uniquely defined by the solution of problem* (20) *at the moment* $\vartheta = 1$*.*

We call equations (20a) and (20b) *the initial infinitesimal layer equations.* Equations (20a) contain function $\Phi(\vartheta)$ and therefore inherit the complete information about the profile of the original non-instantaneous pulse. Due to rescaling $t = \vartheta/n$ (see in (18)), the independent variable ϑ can be regarded to as the *fast time* variable and the pair $(\overline{\boldsymbol{v}},\overline{p}_*)$ can be called a *microscopic initial layer solution*, while t is the *slow time* and the pair $(\boldsymbol{v},p_*)$ is a *macroscopic outer solution*. Thus, when viewed together, system (20)–(21) is the two-scale microscopic-macroscopic problem. Condition (21c) may be fairly interpreted as the *interfacial* condition between the initial microscopic impulsive layer and the macroscopic outer flow. As well, vector-function $\overline{\boldsymbol{v}}(\cdot,1)$ can be regarded to as the "corrected" initial velocity field.

The respective notions of strong solution to problem (20) and weak solutions to problem (21) are as follows.

Definition 2. *Vector-function* $\overline{\boldsymbol{v}} = \overline{\boldsymbol{v}}(\boldsymbol{x},\vartheta)$ *is called a* strong solution *to problem* (20)*, if it satisfies the following requirements:*

- $\overline{\boldsymbol{v}} \in C([0,1];\boldsymbol{H})$, $\quad \partial_\vartheta \overline{\boldsymbol{v}} \in L^\infty(0,1;\boldsymbol{H})$,
- *equation* (20a) *holds a.e. in* $\Omega\times(0,1)$ *with some function*

$$\overline{p}_* \in L^\infty(0,1;W^{1,2}(\Omega)),$$

- *the initial condition* (20c) *holds in the sense of the strong trace in* $\boldsymbol{H}$, *i.e.,*

$$\|\overline{\boldsymbol{v}}(\cdot,\vartheta)-\boldsymbol{v}_0\|_{\boldsymbol{H}} \underset{\vartheta\to 0+}{\longrightarrow} 0. \tag{22}$$

Definition 3. *Vector-function* $\boldsymbol{v}=\boldsymbol{v}(\boldsymbol{x},t)$ *is called a* weak solution *to problem* (21), *if it satisfies the following requirements:*

- $\boldsymbol{v}$ *meets the regularity requirements*

$$\begin{aligned}&\boldsymbol{v}\in L^\infty(0,T;\boldsymbol{H})\cap L^2(0,T;\boldsymbol{V}), \quad \partial_t\boldsymbol{v}\in L^{4/d}(0,T;\boldsymbol{V}^{-1}),\\ &\boldsymbol{v}\in C([0,T];\boldsymbol{H}) \ \text{ if } d=2, \quad \boldsymbol{v}\in C_{\text{weak}}([0,T];\boldsymbol{H}) \ \text{ if } d=3;\end{aligned} \tag{23}$$

- *the integral equality*

$$\begin{aligned}\int_{Q_T}\Big(-\boldsymbol{v}(\boldsymbol{x},t)\cdot\partial_t\boldsymbol{w}(\boldsymbol{x},t)-\big(\boldsymbol{v}(\boldsymbol{x},t)\otimes\boldsymbol{v}(\boldsymbol{x},t)\big):\nabla_x\boldsymbol{w}(\boldsymbol{x},t)\\ +\mu\nabla_x\boldsymbol{v}(\boldsymbol{x},t):\nabla_x\boldsymbol{w}(\boldsymbol{x},t)\Big)d\boldsymbol{x}dt=\int_0^T\big\langle\boldsymbol{f}(\cdot,t),\boldsymbol{w}(\cdot,t)\big\rangle_{\boldsymbol{V}^{-1},\boldsymbol{V}}dt\end{aligned} \tag{24}$$

is valid for all test vector-functions $\boldsymbol{w}\in L^2(0,T;\boldsymbol{V})$ *such that* $\partial_t\boldsymbol{w}\in L^2(0,T;\boldsymbol{H})$ *and* $\boldsymbol{w}=0$ *in a neighborhood of the sections* $\{t=0\}$ *and* $\{t=T\}$*;*

- *in the case* $d=2$, *the initial condition* (21c) *holds in the sense of the strong trace in* $\boldsymbol{H}$, *i.e.*

$$\|\boldsymbol{v}(\cdot,t)-\overline{\boldsymbol{v}}(\cdot,1)\|_{\boldsymbol{H}} \underset{t\to 0+}{\longrightarrow} 0; \tag{25}$$

in the case $d=3$, *the initial condition* (21c) *holds in the sense of the weak trace in* $\boldsymbol{H}$, *i.e.*

$$\int_\Omega\boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{w}(\boldsymbol{x})d\boldsymbol{x}\underset{t\to 0+}{\longrightarrow}\int_\Omega\overline{\boldsymbol{v}}(\boldsymbol{x},1)\cdot\boldsymbol{w}(\boldsymbol{x})d\boldsymbol{x}, \quad \forall\,\boldsymbol{w}\in\boldsymbol{H}. \tag{26}$$

The proof of Theorem 1 is given further in Section 5.

4 Proof of Proposition 2

Estimate (15) follows from the energy inequality (14) in a standard way by virtue of the Cauchy and Cauchy–Schwartz inequalities, conditions (2)–(4), properties (5), and the Grönwall–Bellman lemma.

More certainly, at first, we estimate the right-hand side of (14) to get

$$\begin{aligned}&\frac12\|\boldsymbol{v}_n(\cdot,t)\|^2_{\boldsymbol{H}}+\mu\int_0^t\int_\Omega|\nabla_x\boldsymbol{v}_n(\boldsymbol{x},t')|^2d\boldsymbol{x}dt'\\ &\le\frac12\|\boldsymbol{v}_0\|^2_{\boldsymbol{H}}+\int_0^t\varphi_n(t')\int_\Omega(\boldsymbol{b}(\boldsymbol{x})+\mathbb{B}(\boldsymbol{x})\boldsymbol{v}_n)\cdot\boldsymbol{v}_n d\boldsymbol{x}dt'\\ &\qquad\qquad+\int_0^t\big\langle\boldsymbol{f}(\cdot,t'),\boldsymbol{v}_n(\cdot,t')\big\rangle_{\boldsymbol{V}^{-1},\boldsymbol{V}}dt'\end{aligned}$$

$$\overset{\substack{\text{C.-Schw.}\\ \text{ineq.,}\\ (2)}}{\leq} \frac{1}{2}\|\boldsymbol{v}_0\|^2_{\boldsymbol{H}} + \int_0^t \varphi_n(t') \int_\Omega \left(\frac{M_0^2}{2} + \Big(\frac{1}{2} + M_0 d\Big) |\boldsymbol{v}_n(\boldsymbol{x},t')|^2 \right) d\boldsymbol{x} dt' + \int_0^t \|\boldsymbol{f}(\cdot,t')\|_{\boldsymbol{V}^{-1}} \|\boldsymbol{v}_n(\cdot,t')\|_{\boldsymbol{V}} dt'$$

$$\overset{\substack{\text{Cauchy's}\\ \text{ineq.}}}{\leq} \frac{1}{2}\|\boldsymbol{v}_0\|^2_{\boldsymbol{H}} + \int_0^t \varphi_n(t') \int_\Omega \left(\frac{M_0^2}{2} + \Big(\frac{1}{2} + M_0 d\Big) |\boldsymbol{v}_n(\boldsymbol{x},t')|^2 \right) d\boldsymbol{x} dt' + \frac{1}{2\epsilon}\|\boldsymbol{f}\|^2_{L^2(0,T;\boldsymbol{V}^{-1})} + \frac{\epsilon}{2}\int_0^t \int_\Omega |\boldsymbol{v}_n(\boldsymbol{x},t')|^2 \, d\boldsymbol{x} dt' + \frac{\epsilon}{2}\int_0^t \int_\Omega |\nabla_x \boldsymbol{v}_n(\boldsymbol{x},t')|^2 d\boldsymbol{x} dt'$$

for all $t \in (0,T]$ and $\epsilon > 0$. Taking here $\epsilon := \mu$, multiplying by two, using inequality $(5)_1$ once, and properly re-arranging terms, we arrive at the inequality

$$\|\boldsymbol{v}_n(\cdot,t)\|^2_{\boldsymbol{H}} + \mu \int_0^t \int_\Omega |\nabla_x \boldsymbol{v}_n(\boldsymbol{x},t')|^2 d\boldsymbol{x} dt' \leq \|\boldsymbol{v}_0\|^2_{\boldsymbol{H}} + M_0^2\,|\Omega| + \frac{1}{\mu}\|\boldsymbol{f}\|^2_{L^2(0,T;\boldsymbol{V}^{-1})} + \int_0^t \big((1+2M_0 d)\varphi_n(t') + \mu\big) \|\boldsymbol{v}_n(\cdot,t')\|^2_{\boldsymbol{H}} \, dt', \quad \forall\, t \in (0,T]. \tag{27}$$

At second, we discard the second summand in the left-hand side of (27) and apply the Grönwall–Bellman lemma to the resulting inequality. Thus we establish the estimate

$$\|\boldsymbol{v}_n(\cdot,t)\|^2_{\boldsymbol{H}} \leq \Big(\|\boldsymbol{v}_0\|^2_{\boldsymbol{H}} + M_0^2\,|\Omega| + \frac{1}{\mu}\|\boldsymbol{f}\|^2_{L^2(0,T;\boldsymbol{V}^{-1})} \Big) \exp \int_0^t \big((1+2M_0 d)\varphi_n(t') + \mu\big) dt' \overset{(5)}{\leq} \Big(\|\boldsymbol{v}_0\|^2_{\boldsymbol{H}} + M_0^2\,|\Omega| + \frac{1}{\mu}\|\boldsymbol{f}\|^2_{L^2(0,T;\boldsymbol{V}^{-1})} \Big) \exp\big((1+2M_0 d) + \mu T\big) \overset{\text{def}}{=} M_1 \tag{28}$$

for all $t \in (0,T]$, which yields that

$$\|\boldsymbol{v}_n\|_{L^\infty(0,T;\boldsymbol{H})} \leq \sqrt{M_1}. \tag{29}$$

Further, combining (28) with (27) we easily deduce the bound

$$\|\boldsymbol{v}_n\|_{L^2(0,T;\boldsymbol{V})} \leq M_2, \tag{30}$$

with a positive constant M_2, which depends on d, μ, $|\Omega|$, T, M_0, $\|\boldsymbol{v}_0\|_{\boldsymbol{H}}$, and $\|\boldsymbol{f}\|_{L^2(0,T;\boldsymbol{V}^{-1})}$, and is independent of n.

Joining (29) and (30) we arrive at the estimate

$$\|\boldsymbol{v}_n\|_{L^\infty(0,T;\boldsymbol{H})} + \|\boldsymbol{v}_n\|_{L^2(0,T;\boldsymbol{V})} \le \sqrt{M_1} + M_2. \tag{31}$$

Now, let us turn to derivation of estimate (16).

At first, recall [8, Propos. V.1.3] that the integral equality (12) is equivalent to the integral equality

$$\begin{aligned}
&\big\langle \partial_t \boldsymbol{v}_n(\cdot,t), \boldsymbol{\psi}(\cdot)\big\rangle_{\boldsymbol{V}^{-1},\boldsymbol{V}} \\
&+ \int_\Omega \big(\mathrm{div}_x(\boldsymbol{v}_n(t)\otimes \boldsymbol{v}_n(t))\cdot\boldsymbol{\psi} + \mu\nabla_x \boldsymbol{v}_n(t) : \nabla_x\boldsymbol{\psi}\big)d\boldsymbol{x} = \\
&= \varphi_n(t)\int_\Omega \big(\boldsymbol{b}(\boldsymbol{x}) + \mathbb{B}(\boldsymbol{x})\boldsymbol{v}_n(t)\big)\cdot\boldsymbol{\psi}\,d\boldsymbol{x} + \big\langle \boldsymbol{f}(\cdot,t),\boldsymbol{\psi}(\cdot)\big\rangle_{\boldsymbol{V}^{-1},\boldsymbol{V}}
\end{aligned} \tag{32}$$

for all $\boldsymbol{\psi}\in\boldsymbol{V}$, for a.e. $t\in(0,T)$.

By virtue of the Cauchy–Schwartz and Cauchy–Bunyakovsky inequalities and condition (2), and with the help of Green's formula, for all $\boldsymbol{\psi}\in C_0^1(\overline{\Omega})^d$ from (32) we deduce that

$$\begin{aligned}
&\Big|\big\langle \partial_t \boldsymbol{v}_n(\cdot,t), \boldsymbol{\psi}(\cdot)\big\rangle_{\boldsymbol{V}^{-1},\boldsymbol{V}}\Big| = \\
&= \Big| - \int_\Omega \big(\mathrm{div}_x(\boldsymbol{v}_n(t)\otimes \boldsymbol{v}_n(t))\cdot\boldsymbol{\psi} + \mu\nabla_x \boldsymbol{v}_n(t) : \nabla_x\boldsymbol{\psi}\big)d\boldsymbol{x} \\
&\quad + \varphi_n(t)\int_\Omega \big(\boldsymbol{b}(\boldsymbol{x}) + \mathbb{B}(\boldsymbol{x})\boldsymbol{v}_n(t)\big)\cdot\boldsymbol{\psi}\,d\boldsymbol{x} + \big\langle \boldsymbol{f}(\cdot,t),\boldsymbol{\psi}(\cdot)\big\rangle_{\boldsymbol{V}^{-1},\boldsymbol{V}}\Big| \\
&\overset{\substack{\text{C.-Schw.}\\ \text{ineq.,}\\ \text{Green's}\\ \text{f-la,}\\ (2)}}{\le} \int_\Omega |\boldsymbol{v}_n(t)\otimes\boldsymbol{v}_n(t)||\nabla_x\boldsymbol{\psi}|\,d\boldsymbol{x} + \mu\int_\Omega |\nabla_x\boldsymbol{v}_n(t)||\nabla_x\boldsymbol{\psi}|\,d\boldsymbol{x} \\
&\quad + M_0\varphi_n(t)\int_\Omega (1+d|\boldsymbol{v}_n(t)|)|\boldsymbol{\psi}|\,d\boldsymbol{x} + \|\boldsymbol{f}(\cdot,t)\|_{\boldsymbol{V}^{-1}}\|\boldsymbol{\psi}\|_{\boldsymbol{V}} \\
&\overset{\substack{\text{C.-Bun.}\\ \text{ineq.,}\\ \text{C.-Schw.}\\ \text{ineq.}}}{\le} d\|\boldsymbol{v}_n(\cdot,t)\|_{\boldsymbol{H}}^2\|\nabla_x\boldsymbol{\psi}\|_{C_0(\overline{\Omega})} + \mu|\Omega|^{1/2}\|\nabla_x\boldsymbol{v}_n(\cdot,t)\|_{L^2(\Omega)}\|\nabla_x\boldsymbol{\psi}\|_{C_0(\overline{\Omega})} \\
&\quad + \varphi_n(t)M_0\big(|\Omega| + d|\Omega|^{1/2}\|\boldsymbol{v}_n(\cdot,t)\|_{\boldsymbol{H}}\big)\|\boldsymbol{\psi}\|_{C_0(\overline{\Omega})} \\
&\quad + |\Omega|^{1/2}\|\boldsymbol{f}(\cdot,t)\|_{\boldsymbol{V}^{-1}}\|\boldsymbol{\psi}\|_{C_0^1(\overline{\Omega})} \le \Upsilon_n(t)\|\boldsymbol{\psi}\|_{C_0^1(\overline{\Omega})}
\end{aligned} \tag{33}$$

for a.e. $t\in(0,T)$, where, for the sake of brevity, we denote

$$\begin{aligned}
\Upsilon_n(t) = {}& d\|\boldsymbol{v}_n(\cdot,t)\|_{\boldsymbol{H}}^2 + \mu|\Omega|^{1/2}\|\nabla_x\boldsymbol{v}_n(\cdot,t)\|_{L^2(\Omega)} \\
&+ \varphi_n(t)M_0\big(|\Omega| + d|\Omega|^{1/2}\|\boldsymbol{v}_n(\cdot,t)\|_{\boldsymbol{H}}\big) + |\Omega|^{1/2}\|\boldsymbol{f}(\cdot,t)\|_{\boldsymbol{V}^{-1}}.
\end{aligned}$$

At second, note that, by the Sobolev embedding theorem [26, Ch. I], $\boldsymbol{V}^3$ is compactly embedded in $C_0^1(\overline{\Omega})$, which implies that

$$\|\boldsymbol{\psi}\|_{C_0^1(\overline{\Omega})} \leq C_{Sob}(\Omega)\|\boldsymbol{\psi}\|_{\boldsymbol{V}^3}, \quad \forall\, \boldsymbol{\psi} \in \boldsymbol{V}^3, \tag{34}$$

where $C_{Sob}(\Omega)$ is a constant independent of $\boldsymbol{\psi}$.

Now, combining (33) and (34) we establish the inequality

$$\left|\big\langle \partial_t \boldsymbol{v}_n(\cdot,t), \boldsymbol{\psi}(\cdot)\big\rangle_{\boldsymbol{V}^{-3},\boldsymbol{V}^3}\right| \leq C_{Sob}(\Omega)\Upsilon_n(t)\|\boldsymbol{\psi}\|_{\boldsymbol{V}^3}, \quad \forall\, \boldsymbol{\psi} \in \boldsymbol{V}^3; \tag{35}$$

hence

$$\big\|\partial_t \boldsymbol{v}_n(\cdot,t)\big\|_{\boldsymbol{V}^{-3}} \leq C_{Sob}(\Omega)\Upsilon_n(t) \quad \text{for a.e. } t \in (0,T). \tag{36}$$

Finally, by virtue of (5), (6), (29), and (30) we conclude that there is a positive constant M_3, independent of n, such that

$$\int_0^T \Upsilon_n(t)dt \leq M_3, \quad \forall\, n \geq n_0,$$

which along with (36) implies that

$$\big\|\partial_t \boldsymbol{v}_n(\cdot,t)\big\|_{L^1(0,T;\boldsymbol{V}^{-3})} \leq C_{Sob}(\Omega)M_3. \tag{37}$$

Estimates (31) and (37) now yield estimates (15) and (16) with

$$C_0 = \max\big\{\sqrt{M_1} + M_2, C_{Sob}(\Omega)M_3\big\}.$$

Proposition 2 is proved.

5 Proof of Theorem 1

5.1. Relative compactness of the family $\{\boldsymbol{v}_n\}_{n\geq n_0}$. Let us recall the following compactness result.

The Aubin–Lions–Simon Lemma. ([24, Corol. 4]) *Assume X, B and Y are three Banach spaces such that $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$, i.e. X is compactly embedded in B and B is continuously embedded in Y. Then, the following assertion holds true:*

If some set $\mathcal{F}$ is bounded in $L^p(0,T;X)$, where $1 \leq p < +\infty$, and set $\partial\mathcal{F}/\partial t \stackrel{\text{def}}{=} \big\{\partial_t\phi,\ \phi \in \mathcal{F}\big\}$ is bounded in $L^1(0,T;Y)$ then $\mathcal{F}$ is relatively compact in $L^p(0,T;B)$.

Proposition 2, the Rellich theorem, the Aubin–Lions–Simon lemma, and the Alaoglu theorem imply that the assertion 1 of Theorem 1 holds true. Indeed, $X := \boldsymbol{V}$ is compactly embedded in $B := \boldsymbol{H}$ by the Rellich theorem; hence, according to (15) and (16), the set $\mathcal{F} := \{\boldsymbol{v}_n\}_{n\geq n_0}$ satisfies the hypotheses in the Aubin–Lions–Simon lemma with $p := 2$ and $Y := \boldsymbol{V}^{-3}$. Therefore, by the Aubin–Lions–Simon lemma, the set $\{\boldsymbol{v}_n\}_{n\geq n_0}$ is relatively compact in $L^2(0,T;\boldsymbol{H})$. Bound (15) also implies that, by the Alaoglu theorem, $\{\boldsymbol{v}_n\}_{n\geq n_0}$ is relatively weakly compact in $L^2(0,T;\boldsymbol{V})$ and relatively weakly* compact in $L^\infty(0,T;\boldsymbol{H})$. Due to these properties, there exist a subsequence from $\{\boldsymbol{v}_n\}_{n\geq n_0}$ and a limit vector-function $\boldsymbol{v} \in L^2(0,T;\boldsymbol{V}) \cap$

$L^\infty(0,T;\boldsymbol{H})$ satisfying the limiting relation (17). This completes the proof of assertion 1 of Theorem 1.

5.2. Rescaling and shift. Following the previously carefully developed procedure (see, for example, [3, 4, 5, 16, 17, 18]), let us fulfill some preliminary considerations before we turn to the limiting passage as $n \to +\infty$ in (12).

Assuming that the test vector-function $\boldsymbol{w}$ in the integral equality (12) vanishes in a neighborhood of the section $\{t = T\}$, let us integrate the first summand in the left-hand side of (12) in t by parts, apply Green's formula in $\boldsymbol{x}$ to the convective term, and write out the resulting equality in an expanded form, separating the integrals over segments $(0, 1/n)$ and $(1/n, T)$ from each other:

$$\begin{aligned}
&\int_0^{1/n}\int_\Omega \big(-\boldsymbol{v}_n \cdot \partial_t \boldsymbol{w} - (\boldsymbol{v}_n \otimes \boldsymbol{v}_n) : \nabla_x \boldsymbol{w} + \mu \nabla_x \boldsymbol{v}_n : \nabla_x \boldsymbol{w} \\
&\qquad\qquad - n\Phi(nt)(\boldsymbol{b}(\boldsymbol{x}) + \mathbb{B}(\boldsymbol{x})\boldsymbol{v}_n) \cdot \boldsymbol{w}\big)\, d\boldsymbol{x}dt \\
&- \int_0^{1/n} \big\langle \boldsymbol{f}(\cdot,t), \boldsymbol{w}(\cdot,t)\big\rangle_{\boldsymbol{V}^{-1},\boldsymbol{V}} dt - \int_\Omega \boldsymbol{v}_0(\boldsymbol{x}) \cdot \boldsymbol{w}(\boldsymbol{x},0)\, d\boldsymbol{x} \\
&+ \int_{1/n}^T \int_\Omega \big(-\boldsymbol{v}_n \cdot \partial_t \boldsymbol{w} - (\boldsymbol{v}_n \otimes \boldsymbol{v}_n) : \nabla_x \boldsymbol{w} + \mu \nabla_x \boldsymbol{v}_n : \nabla_x \boldsymbol{w}\big)\, d\boldsymbol{x}dt \\
&\qquad\qquad - \int_{1/n}^T \big\langle \boldsymbol{f}(\cdot,t), \boldsymbol{w}(\cdot,t)\big\rangle_{\boldsymbol{V}^{-1},\boldsymbol{V}} dt = 0,
\end{aligned} \tag{38}$$

where the fact that the support of the function $t \mapsto n\Phi(nt)$ lays in $[0, 1/n]$ is taken into account. In (38), we change the independent variable t and the sought vector-function $\boldsymbol{v}_n$ on the segments $\{0 \le t \le 1/n\}$ and $\{1/n < t \le T\}$ as follows.

On $(1/n, T]$ we shift the timescale backwards and take

$$\widetilde{t} := t - 1/n, \quad \widetilde{\boldsymbol{v}}_n(\boldsymbol{x},\widetilde{t}) := \boldsymbol{v}_n(\boldsymbol{x},t) \equiv \boldsymbol{v}_n(\boldsymbol{x},\widetilde{t}+1/n) \quad \text{for } t \in (1/n, T]. \tag{39}$$

Note that $\widetilde{t} \in (0, T - 1/n]$, $dt = d\widetilde{t}$, $\partial_t = \partial_{\widetilde{t}}$, and $t = \widetilde{t} + 1/n$. Further, on $[0, 1/n]$ we stretch the timescale and take

$$\vartheta := nt, \quad \overline{\boldsymbol{v}}_n(\boldsymbol{x},\vartheta) := \boldsymbol{v}_n(\boldsymbol{x},t) \equiv \boldsymbol{v}_n(\boldsymbol{x},n^{-1}\vartheta) \quad \text{for } t \in [0, 1/n]. \tag{40}$$

Note that $\vartheta \in [0,1]$, $dt = n^{-1}d\vartheta$, $\partial_t = n\partial_\vartheta$, and $t = n^{-1}\vartheta$. Thus, (38) takes the form

$$\begin{aligned}
\int_0^1 \int_\Omega \big(&-\overline{\boldsymbol{v}}_n(\boldsymbol{x},\vartheta) \cdot \partial_\vartheta \boldsymbol{w}(\boldsymbol{x},n^{-1}\vartheta) \\
&- n^{-1}(\overline{\boldsymbol{v}}_n(\boldsymbol{x},\vartheta) \otimes \overline{\boldsymbol{v}}_n(\boldsymbol{x},\vartheta)) : \nabla_x \boldsymbol{w}(\boldsymbol{x},n^{-1}\vartheta) \\
&+ n^{-1}\mu \nabla_x \overline{\boldsymbol{v}}_n(\boldsymbol{x},\vartheta) : \nabla_x \boldsymbol{w}(\boldsymbol{x},n^{-1}\vartheta) \\
&\qquad - \Phi(\vartheta)(\boldsymbol{b}(\boldsymbol{x}) + \mathbb{B}(\boldsymbol{x})\overline{\boldsymbol{v}}_n(\boldsymbol{x},\vartheta)) \cdot \boldsymbol{w}(\boldsymbol{x},n^{-1}\vartheta)\big)\, d\boldsymbol{x}d\vartheta
\end{aligned}$$

$$
\begin{aligned}
&- n^{-1} \int_0^1 \big\langle \boldsymbol{f}(\cdot, n^{-1}\vartheta), \boldsymbol{w}(\cdot, n^{-1}\vartheta)\big\rangle_{\boldsymbol{V}^{-1},\boldsymbol{V}} d\vartheta - \int_\Omega \boldsymbol{v}_0(\boldsymbol{x}) \cdot \boldsymbol{w}(\boldsymbol{x}, 0)\, d\boldsymbol{x} \\
&+ \int_0^{T-1/n} \int_\Omega \big(-\widetilde{\boldsymbol{v}}_n(\boldsymbol{x}, \widetilde{t}) \cdot \partial_{\widetilde{t}} \boldsymbol{w}(\boldsymbol{x}, \widetilde{t} + 1/n) \\
&\qquad - (\widetilde{\boldsymbol{v}}_n(\boldsymbol{x}, \widetilde{t}) \otimes \widetilde{\boldsymbol{v}}_n(\boldsymbol{x}, \widetilde{t})) : \nabla_x \boldsymbol{w}(\boldsymbol{x}, \widetilde{t} + 1/n) \\
&\qquad + \mu \nabla_x \widetilde{\boldsymbol{v}}_n(\boldsymbol{x}, \widetilde{t}) : \nabla_x \boldsymbol{w}(\boldsymbol{x}, \widetilde{t} + 1/n) \big)\, d\boldsymbol{x} d\widetilde{t} \\
&- \int_0^{T-1/n} \big\langle \boldsymbol{f}(\cdot, \widetilde{t} + 1/n), \boldsymbol{w}(\cdot, \widetilde{t} + 1/n) \big\rangle_{\boldsymbol{V}^{-1},\boldsymbol{V}}\, d\widetilde{t} = 0. \qquad (41)
\end{aligned}
$$

In accord with the further limiting passage as $n \to +\infty$, in (41) we take the test vector-function $\boldsymbol{w} = \boldsymbol{w}_n(\boldsymbol{x}, t)$ depending on n in the following form:

$$
\boldsymbol{w}_n(\boldsymbol{x}, t) = \begin{cases} \overline{\boldsymbol{w}}(\boldsymbol{x}, \vartheta) \equiv \overline{\boldsymbol{w}}(\boldsymbol{x}, nt) & \text{for } t \in [0, 1/n], \text{ i.e., for } \vartheta \in [0, 1]; \\ \widetilde{\boldsymbol{w}}(\boldsymbol{x}, \widetilde{t}) \equiv \widetilde{\boldsymbol{w}}(\boldsymbol{x}, t - 1/n) & \text{for } t \in (1/n, T], \\ & \text{i.e., for } \widetilde{t} \in (0, T - 1/n], \end{cases} \qquad (42)
$$

where $\overline{\boldsymbol{w}} = \overline{\boldsymbol{w}}(\boldsymbol{x}, \vartheta)$ and $\widetilde{\boldsymbol{w}} = \widetilde{\boldsymbol{w}}(\boldsymbol{x}, \widetilde{t})$ are arbitrary smooth test vector-functions defined on $\overline{\Omega} \times [0, 1]$ and $\overline{\Omega} \times [0, T]$, respectively, such that $\overline{\boldsymbol{w}} = \widetilde{\boldsymbol{w}} \equiv 0$ in a neighborhood of $\partial\Omega$, $\widetilde{\boldsymbol{w}} \equiv 0$ in a neighborhood of the plane $\{\widetilde{t} = T\}$, and the matching condition

$$
\overline{\boldsymbol{w}}(\boldsymbol{x}, 1 - 0) = \widetilde{\boldsymbol{w}}(\boldsymbol{x}, 0+) \qquad (43)
$$

holds. We notice that condition (43) yields that the weak derivative $\partial_t \boldsymbol{w}_n$ is bounded in Q_T, which implies that $\boldsymbol{w}_n \in C([0, T]; \boldsymbol{V})$ and $\partial_t \boldsymbol{w}_n \in L^2(0, T; \boldsymbol{H})$ and therefore $\boldsymbol{w}_n$ is an admissible test vector-function for (38) and (41). Inserting (42) into (41), we get

$$
\begin{aligned}
&\int_0^1 \int_\Omega \big(-\overline{\boldsymbol{v}}_n(\boldsymbol{x}, \vartheta) \cdot \partial_\vartheta \overline{\boldsymbol{w}}(\boldsymbol{x}, \vartheta) - n^{-1} (\overline{\boldsymbol{v}}_n(\boldsymbol{x}, \vartheta) \otimes \overline{\boldsymbol{v}}_n(\boldsymbol{x}, \vartheta)) : \nabla_x \overline{\boldsymbol{w}}(\boldsymbol{x}, \vartheta) \\
&\qquad + n^{-1} \mu \nabla_x \overline{\boldsymbol{v}}_n(\boldsymbol{x}, \vartheta) : \nabla_x \overline{\boldsymbol{w}}(\boldsymbol{x}, \vartheta) \\
&\qquad - \Phi(\vartheta)(\boldsymbol{b}(\boldsymbol{x}) + \mathbb{B}(\boldsymbol{x}) \overline{\boldsymbol{v}}_n(\boldsymbol{x}, \vartheta)) \cdot \overline{\boldsymbol{w}}(\boldsymbol{x}, \vartheta) \big)\, d\boldsymbol{x} d\vartheta \\
&- n^{-1} \int_0^1 \big\langle \boldsymbol{f}(\cdot, n^{-1}\vartheta), \overline{\boldsymbol{w}}(\cdot, \vartheta) \big\rangle_{\boldsymbol{V}^{-1},\boldsymbol{V}} d\vartheta - \int_\Omega \boldsymbol{v}_0(\boldsymbol{x}) \cdot \overline{\boldsymbol{w}}(\boldsymbol{x}, 0)\, d\boldsymbol{x} \\
&+ \int_0^{T-1/n} \int_\Omega \big(-\widetilde{\boldsymbol{v}}_n(\boldsymbol{x}, \widetilde{t}) \cdot \partial_{\widetilde{t}} \widetilde{\boldsymbol{w}}(\boldsymbol{x}, \widetilde{t}) - (\widetilde{\boldsymbol{v}}_n(\boldsymbol{x}, \widetilde{t}) \otimes \widetilde{\boldsymbol{v}}_n(\boldsymbol{x}, \widetilde{t})) : \nabla_x \widetilde{\boldsymbol{w}}(\boldsymbol{x}, \widetilde{t}) \\
&\qquad + \mu \nabla_x \widetilde{\boldsymbol{v}}_n(\boldsymbol{x}, \widetilde{t}) : \nabla_x \widetilde{\boldsymbol{w}}(\boldsymbol{x}, \widetilde{t}) \big)\, d\boldsymbol{x} d\widetilde{t} \\
&- \int_0^{T-1/n} \big\langle \boldsymbol{f}(\cdot, \widetilde{t} + 1/n), \widetilde{\boldsymbol{w}}(\cdot, \widetilde{t}) \big\rangle_{\boldsymbol{V}^{-1},\boldsymbol{V}} d\widetilde{t} = 0. \qquad (44)
\end{aligned}
$$

Furthermore, we notice that

$$\overline{\boldsymbol{v}}_n(\boldsymbol{x},1-0)=\widetilde{\boldsymbol{v}}_n(\boldsymbol{x},0+)\quad\text{in }\Omega \tag{45}$$

due to (39), (40) and the regularity properties of $\boldsymbol{v}_n$ (see inclusions (11) in Definition 1). The rest of the proof of Theorem 1 is based on a thorough study of (44) with account of (45).

5.3. Limiting passage in the family $\{\overline{\boldsymbol{v}}_n\}_{n\geq n_0}$. The initial infinitesimal layer equations. Applying the shift and rescaling (i.e. transformations (39) and (40)) in estimates (15) and (16) and discarding the nonnegative expressions containing $\widetilde{\boldsymbol{v}}_n$, we derive the following estimates for the family $\{\overline{\boldsymbol{v}}_n\}_{n\geq n_0}$:

$$\|\overline{\boldsymbol{v}}_n\|_{L^\infty(0,1;\boldsymbol{H})}+n^{-1/2}\|\overline{\boldsymbol{v}}_n\|_{L^2(0,1;\boldsymbol{V})}\leq\|\boldsymbol{v}_n\|_{L^\infty(0,T;\boldsymbol{H})}+\|\boldsymbol{v}_n\|_{L^2(0,T;\boldsymbol{V})}\leq C_0, \tag{46}$$

$$\|\partial_\vartheta\overline{\boldsymbol{v}}_n\|_{L^1(0,1;\boldsymbol{V}^{-3})}\leq\|\partial_t\boldsymbol{v}_n\|_{L^1(0,T;\boldsymbol{V}^{-3})}\leq C_0, \tag{47}$$

where C_0 is the same constant, as in (15) and (16).

Estimate (46) implies that the family $\{\overline{\boldsymbol{v}}_n\}_{n\geq n_0}$ is uniformly bounded in $L^\infty(0,1;\boldsymbol{H})$. Due to this, assertion 2 of Theorem 1 follows immediately from the Alaoglu theorem. Moreover, the limit vector-function $\overline{\boldsymbol{v}}=\mathrm{w}^\star\text{-}\lim\limits_{n\to+\infty}\overline{\boldsymbol{v}}_n$ admits the bound

$$\|\overline{\boldsymbol{v}}\|_{L^\infty(0,1;\boldsymbol{H})}\leq C_0. \tag{48}$$

Further, taking $\widetilde{\boldsymbol{w}}\equiv 0$ in (44), we get

$$\begin{aligned}\int_0^1\int_\Omega\big(&-\overline{\boldsymbol{v}}_n(\boldsymbol{x},\vartheta)\cdot\partial_\vartheta\overline{\boldsymbol{w}}(\boldsymbol{x},\vartheta)-n^{-1}(\overline{\boldsymbol{v}}_n(\boldsymbol{x},\vartheta)\otimes\overline{\boldsymbol{v}}_n(\boldsymbol{x},\vartheta)):\nabla_x\overline{\boldsymbol{w}}(\boldsymbol{x},\vartheta)\\&+n^{-1}\mu\nabla_x\overline{\boldsymbol{v}}_n(\boldsymbol{x},\vartheta):\nabla_x\overline{\boldsymbol{w}}(\boldsymbol{x},\vartheta)\\&-\Phi(\vartheta)(\boldsymbol{b}(\boldsymbol{x})+\mathbb{B}(\boldsymbol{x})\overline{\boldsymbol{v}}_n(x,\vartheta))\cdot\overline{\boldsymbol{w}}(\boldsymbol{x},\vartheta)\big)d\boldsymbol{x}d\vartheta\\-n^{-1}\int_0^1&\big\langle\boldsymbol{f}(\cdot,n^{-1}\vartheta),\overline{\boldsymbol{w}}(\cdot,\vartheta)\big\rangle_{\boldsymbol{V}^{-1},\boldsymbol{V}}d\vartheta-\int_\Omega\boldsymbol{v}_0(\boldsymbol{x})\cdot\overline{\boldsymbol{w}}(\boldsymbol{x},0)d\boldsymbol{x}=0\end{aligned} \tag{49}$$

for all test vector-functions $\overline{\boldsymbol{w}}$ satisfying the conditions imposed above for (42) and vanishing in a neighborhood of the plane $\{\vartheta=1\}$.

Due to the elementary inequality $FG\leq(F^2+G^2)/2$ $(\forall\,F,G\in\mathbb{R})$, and estimate (46), we have

$$\begin{aligned}&\left|\int_0^1\int_\Omega n^{-1}(\overline{\boldsymbol{v}}_n(\boldsymbol{x},\vartheta)\otimes\overline{\boldsymbol{v}}_n(\boldsymbol{x},\vartheta)):\nabla_x\overline{\boldsymbol{w}}(\boldsymbol{x},\vartheta)d\boldsymbol{x}d\vartheta\right|\\&\leq\frac{d}{2n}\|\overline{\boldsymbol{v}}_n\|^2_{L^\infty(0,1;\boldsymbol{H})}\|\nabla_x\overline{\boldsymbol{w}}\|_{C(\overline{\Omega}\times[0,1])}\leq\frac{d}{2n}C_0^2\|\nabla_x\overline{\boldsymbol{w}}\|_{C(\overline{\Omega}\times[0,1])}\underset{n\to+\infty}{\longrightarrow}0,\end{aligned} \tag{50}$$

i.e. the convective term in (49) converges to zero as $n \to +\infty$. Due to the Cauchy–Bunyakovsky inequality and estimate (46), we establish

$$\left| \int_0^1 \int_\Omega n^{-1} \mu \nabla_x \overline{\boldsymbol{v}}_n(\boldsymbol{x}, \vartheta) : \nabla_x \overline{\boldsymbol{w}}(\boldsymbol{x}, \vartheta) d\boldsymbol{x} d\vartheta \right| \\ \leq \frac{\mu}{n} \|\nabla_x \overline{\boldsymbol{v}}_n\|_{L^2(0,1;L^2(\Omega)^{d\times d})} \|\nabla_x \overline{\boldsymbol{w}}\|_{L^2(0,1;L^2(\Omega)^{d\times d})} \\ \leq \frac{\mu}{\sqrt{n}} C_0 \|\nabla_x \overline{\boldsymbol{w}}\|_{L^2(0,1;L^2(\Omega)^{d\times d})} \underset{n\to+\infty}{\longrightarrow} 0, \tag{51}$$

i.e. the viscous dissipative term in (49) also converges to zero as $n \to +\infty$. Analogously,

$$\left| n^{-1} \int_0^1 \left\langle \boldsymbol{f}(\cdot, n^{-1}\vartheta), \overline{\boldsymbol{w}}(\cdot, \vartheta) \right\rangle_{\boldsymbol{V}^{-1}, \boldsymbol{V}} d\vartheta \right| \\ \leq \frac{1}{n} \int_0^1 \|\boldsymbol{f}(\cdot, n^{-1}\vartheta)\|_{\boldsymbol{V}^{-1}} \|\overline{\boldsymbol{w}}(\cdot, \vartheta)\|_{\boldsymbol{V}} d\vartheta \\ \leq \frac{1}{n} \|\boldsymbol{f}\|_{C([0,T];\boldsymbol{V}^{-1})} \|\overline{\boldsymbol{w}}\|_{L^1(0,1;\boldsymbol{V})} \underset{n\to+\infty}{\longrightarrow} 0. \tag{52}$$

By virtue of (19), for the remaining terms in (49) we have

$$\int_0^1 \int_\Omega \big(-\overline{\boldsymbol{v}}_n(\boldsymbol{x}, \vartheta) \cdot \partial_\vartheta \overline{\boldsymbol{w}}(\boldsymbol{x}, \vartheta) - \Phi(\vartheta)(\boldsymbol{b}(\boldsymbol{x}) + \mathbb{B}(\boldsymbol{x}) \overline{\boldsymbol{v}}_n(x, \vartheta)) \cdot \overline{\boldsymbol{w}}(\boldsymbol{x}, \vartheta) \big) d\boldsymbol{x} d\vartheta \\ - \int_\Omega \boldsymbol{v}_0(\boldsymbol{x}) \cdot \overline{\boldsymbol{w}}(\boldsymbol{x}, 0) d\boldsymbol{x} \underset{n\to+\infty}{\longrightarrow} \\ \int_0^1 \int_\Omega \big(-\overline{\boldsymbol{v}}(\boldsymbol{x}, \vartheta) \cdot \partial_\vartheta \overline{\boldsymbol{w}}(\boldsymbol{x}, \vartheta) - \Phi(\vartheta)(\boldsymbol{b}(\boldsymbol{x}) + \mathbb{B}(\boldsymbol{x}) \overline{\boldsymbol{v}}(x, \vartheta)) \cdot \overline{\boldsymbol{w}}(\boldsymbol{x}, \vartheta) \big) d\boldsymbol{x} d\vartheta \\ - \int_\Omega \boldsymbol{v}_0(\boldsymbol{x}) \cdot \overline{\boldsymbol{w}}(\boldsymbol{x}, 0) d\boldsymbol{x}. \tag{53}$$

Now, collecting the results (50)–(53), from (49) we deduce the integral equality

$$\int_0^1 \int_\Omega \big(-\overline{\boldsymbol{v}}(\boldsymbol{x}, \vartheta) \cdot \partial_\vartheta \overline{\boldsymbol{w}}(\boldsymbol{x}, \vartheta) - \Phi(\vartheta)(\boldsymbol{b}(\boldsymbol{x}) + \mathbb{B}(\boldsymbol{x}) \overline{\boldsymbol{v}}(x, \vartheta)) \cdot \overline{\boldsymbol{w}}(\boldsymbol{x}, \vartheta) \big) d\boldsymbol{x} d\vartheta \\ - \int_\Omega \boldsymbol{v}_0(\boldsymbol{x}) \cdot \overline{\boldsymbol{w}}(\boldsymbol{x}, 0) d\boldsymbol{x} = 0 \tag{54}$$

for all test vector-functions $\overline{\boldsymbol{w}}$ satisfying the above imposed conditions.

By virtue of (2), (4) and (48), using a standard procedure similar to derivation of the bound (37), from (54) we easily deduce the estimate

$$\|\partial_\vartheta \overline{\boldsymbol{v}}\|_{L^\infty(0,1;\boldsymbol{H})} \leq M_0 \|\Phi\|_{C[0,1]} \big(1 + d\|\overline{\boldsymbol{v}}\|_{L^\infty(0,1;\boldsymbol{H})}\big) \leq M_0 \|\Phi\|_{C[0,1]} (1 + dC_0); \tag{55}$$

hence $\partial_\vartheta \overline{\boldsymbol{v}} \in L^\infty(0,1;\boldsymbol{H})$, which along with inclusion $\overline{\boldsymbol{v}} \in L^\infty(0,1;\boldsymbol{H})$ by [24, Lem. 4] gives that $\overline{\boldsymbol{v}} \in C([0,1];\boldsymbol{H})$.

Finally, due to these regularity properties of $\overline{\boldsymbol{v}}$, the sufficient arbitrariness of $\overline{\boldsymbol{w}}$, and the Weyl decomposition

$$L^2(\Omega)^d = \boldsymbol{H} \oplus \nabla W^{1,2}(\Omega),$$

the integral equality (54) implies that equation (20a) holds a.e. in $\Omega \times (0,1)$ with some scalar function $\overline{p}_*$ and the initial condition (20c) holds in the sense of the strong trace in $\boldsymbol{H}$. That is, $\overline{\boldsymbol{v}}$ is a strong solution to problem (20).

5.4. Uniqueness of $\overline{\boldsymbol{v}}$. The following proposition holds true.

Proposition 3. *Assume that the data Ω, $\boldsymbol{v}_0$, Φ, $\boldsymbol{b}$, $\mathbb{B}$ satisfy the conditions imposed in Section 1. Then the strong solution $\overline{\boldsymbol{v}}$ to problem (20) is unique.*

Proof. Let $\overline{\boldsymbol{v}}_1$ and $\overline{\boldsymbol{v}}_2$ be two strong solutions of problem (20), corresponding to the same given data Φ, $\boldsymbol{b}$, $\mathbb{B}$, and $\boldsymbol{v}_0$. Denote $\overline{\boldsymbol{v}}_* := \overline{\boldsymbol{v}}_2 - \overline{\boldsymbol{v}}_1$. Subtracting the equation (20a) with $\overline{\boldsymbol{v}}_1$ from the equation (20a) with $\overline{\boldsymbol{v}}_2$, multiplying the resulting equation by $\overline{\boldsymbol{v}}_*$, integrating over Ω, employing condition (2) for b_{ij}, and applying the Cauchy–Bunyakovsky inequality, we get

$$\frac{d}{d\vartheta}\|\overline{\boldsymbol{v}}_*(\cdot,\vartheta)\|^2_{\boldsymbol{H}} = \Phi(\vartheta)\int_\Omega \mathbb{B}(\boldsymbol{x})\overline{\boldsymbol{v}}_*(\boldsymbol{x},\vartheta)\cdot\overline{\boldsymbol{v}}_*(\boldsymbol{x},\vartheta)d\boldsymbol{x} \le dM_0\Phi(\vartheta)\|\overline{\boldsymbol{v}}_*(\cdot,\vartheta)\|^2_{\boldsymbol{H}}, \quad \forall\,\vartheta\in[0,1].$$

By Grönwall's lemma, from this inequality we deduce that

$$\|\overline{\boldsymbol{v}}_*(\cdot,\vartheta)\|^2_{\boldsymbol{H}} \le \|\overline{\boldsymbol{v}}_*(\cdot,0)\|^2_{\boldsymbol{H}}\exp\Big\{dM_0\int_0^\vartheta \Phi(\vartheta')d\vartheta'\Big\} = 0, \quad \forall\,\vartheta\in[0,1],$$

since $\overline{\boldsymbol{v}}_*(\cdot,0) = \boldsymbol{v}_0 - \boldsymbol{v}_0 = 0$. Thus $\overline{\boldsymbol{v}}_* \equiv 0$; hence $\overline{\boldsymbol{v}}_1 = \overline{\boldsymbol{v}}_2$ in $\Omega\times(0,1)$. Proposition 3 is proved. □

Having proved this proposition, we have completed the justification of assertion 3 of Theorem 1.

5.5. Limiting passage in the family $\{\boldsymbol{v}_n\}_{n\ge n_0}$. Equations of the outer flow. Introduce into consideration the characteristic function of the segment $\{0<\widetilde{t}<T-1/n\}$:

$$\theta_n(\widetilde{t}) = \begin{cases} 1 & \text{for} \quad 0<\widetilde{t}<T-1/n, \\ 0 & \text{for} \quad \widetilde{t}\notin(0,T-1/n). \end{cases}$$

Taking $\overline{\boldsymbol{w}}\equiv 0$ in (44), we get

$$\int_0^T\int_\Omega \theta_n(\widetilde{t})\big(-\widetilde{\boldsymbol{v}}_n(\boldsymbol{x},\widetilde{t})\cdot\partial_{\widetilde{t}}\widetilde{\boldsymbol{w}}(\boldsymbol{x},\widetilde{t}) - (\widetilde{\boldsymbol{v}}_n(\boldsymbol{x},\widetilde{t})\otimes\widetilde{\boldsymbol{v}}_n(\boldsymbol{x},\widetilde{t})):\nabla_x\widetilde{\boldsymbol{w}}(\boldsymbol{x},\widetilde{t}) + \mu\nabla_x\widetilde{\boldsymbol{v}}_n(\boldsymbol{x},\widetilde{t}):\nabla_x\widetilde{\boldsymbol{w}}(\boldsymbol{x},\widetilde{t})\big)\,d\boldsymbol{x}d\widetilde{t}$$
$$-\int_0^T\theta_n(\widetilde{t})\big\langle\boldsymbol{f}(\cdot,\widetilde{t}+1/n),\widetilde{\boldsymbol{w}}(\cdot,\widetilde{t})\big\rangle_{\boldsymbol{V}^{-1},\boldsymbol{V}}d\widetilde{t} = 0 \tag{56}$$

for all test vector-functions $\widetilde{\boldsymbol{w}}$ satisfying the conditions imposed above for (42) and vanishing in a neighborhood of the plane $\{\widetilde{t}=0\}$.

Similarly to estimates (46) and (47), using the shift and rescaling we derive the following uniform in n estimates for the family $\{\widetilde{\boldsymbol{v}}_n\}_{n\ge n_0}$:

$$\|\widetilde{\boldsymbol{v}}_n\|_{L^\infty(0,T-1/n;\boldsymbol{H})} + \|\widetilde{\boldsymbol{v}}_n\|_{L^2(0,T-1/n;\boldsymbol{V})} \le C_0, \tag{57}$$

$$\|\partial_{\widetilde{t}}\widetilde{\boldsymbol{v}}_n\|_{L^1(0,T-1/n;\boldsymbol{V}^{-3})} \le C_0, \tag{58}$$

where C_0 is the same constant, as in estimates (15), (16), (46), and (47). Due to (57), (58), the Aubin–Lions–Simon lemma, the Alaoglu theorem, and the limiting relation

$$\theta_n \underset{n\to+\infty}{\longrightarrow} 1 \quad \text{weakly}^\star \text{ in } L^\infty(0,T) \text{ and strongly in } L^r(0,T) \ \ \forall\, r\in[1,+\infty), \tag{59}$$

there exist a subsequence from $\{\widetilde{\boldsymbol{v}}_n\}$ and a limit vector-function

$$\widetilde{\boldsymbol{v}} \in L^\infty(0,T;\boldsymbol{H})\cap L^2(0,T;\boldsymbol{V})$$

such that

$$\theta_n\widetilde{\boldsymbol{v}}_n \underset{n\to+\infty}{\longrightarrow} \widetilde{\boldsymbol{v}} \quad \text{weakly}^\star \text{ in } L^\infty(0,T;\boldsymbol{H}),\ \text{strongly in } L^{2-\nu}(0,T;\boldsymbol{H}) \text{ and weakly in } L^{2-\nu}(0,T;\boldsymbol{V}),\ \forall\,\nu\in(0,1]. \tag{60}$$

In turn, the strong and weak* limiting relations in (60) yield that

$$\theta_n\widetilde{\boldsymbol{v}}_n\otimes\widetilde{\boldsymbol{v}}_n \equiv \theta_n\widetilde{\boldsymbol{v}}_n\otimes(\theta_n\widetilde{\boldsymbol{v}}_n) \underset{n\to+\infty}{\longrightarrow} \widetilde{\boldsymbol{v}}\otimes\widetilde{\boldsymbol{v}} \quad \text{weakly in } L^{2-\nu}\big(0,T;L^1(\Omega)^{d\times d}\big). \tag{61}$$

Further, due to representations (39), estimate (16), and the elementary finite increment formula (see in [24, Lem. 4]), we have

$$\begin{aligned}
&\int_0^{T-1/n} \|\widetilde{\boldsymbol{v}}_n(\cdot,\widetilde{t}) - \boldsymbol{v}_n(\cdot,\widetilde{t})\|_{\boldsymbol{V}^{-3}}\,d\widetilde{t} = \\
&\overset{(39)}{=} \int_0^{T-1/n} \|\boldsymbol{v}_n(\cdot,\widetilde{t}+1/n) - \boldsymbol{v}_n(\cdot,\widetilde{t})\|_{\boldsymbol{V}^{-3}}\,d\widetilde{t} \\
&\overset{[24,\text{ Lem. }4]}{\le} n^{-1}\int_0^T \|\partial_{\widetilde{t}}\boldsymbol{v}_n(\cdot,\widetilde{t})\|_{\boldsymbol{V}^{-3}}\,d\widetilde{t} \overset{(16)}{\le} n^{-1}C_0 \underset{n\to+\infty}{\longrightarrow} 0.
\end{aligned} \tag{62}$$

Since $\boldsymbol{H}$ is compactly embedded in $\boldsymbol{V}^{-3}$, the strong limiting relations in (17) and (60) imply that

$$\int_0^T \|\boldsymbol{v}_n(\cdot,\widetilde{t}) - \boldsymbol{v}(\cdot,\widetilde{t})\|_{\boldsymbol{V}^{-3}}\,d\widetilde{t} \underset{n\to+\infty}{\longrightarrow} 0, \tag{63}$$

$$\int_0^{T-1/n_1} \|\widetilde{\boldsymbol{v}}_n(\cdot,\widetilde{t}) - \widetilde{\boldsymbol{v}}(\cdot,\widetilde{t})\|_{\boldsymbol{V}^{-3}}\,d\widetilde{t} \underset{n\to+\infty}{\longrightarrow} 0, \quad \forall\, n_1\ge n_0. \tag{64}$$

Combining (62), (63) and (64) and using arbitrariness of n_1, by the triangle inequality we deduce that $\widetilde{\boldsymbol{v}} = \boldsymbol{v}$ in $L^1(0,T;\boldsymbol{V}^{-3})$. Since both $\widetilde{\boldsymbol{v}}$ and $\boldsymbol{v}$ belong to $L^2(0,T;\boldsymbol{H})$, this implies that

$$\widetilde{\boldsymbol{v}}(\boldsymbol{x},\widetilde{t}) = \boldsymbol{v}(\boldsymbol{x},\widetilde{t}) \quad \text{for a.e. } (\boldsymbol{x},\widetilde{t})\in Q_T. \tag{65}$$

Now, using relations $(6)_1$, (59)–(61) and (65) and re-denoting $t := \widetilde{t}$, we pass to the limit in (56) as $n \to +\infty$ and by this derive the integral equality

$$\int_0^T \int_\Omega \big(-\boldsymbol{v}(\boldsymbol{x},t)\cdot\partial_t\widetilde{\boldsymbol{w}}(\boldsymbol{x},t) - (\boldsymbol{v}(\boldsymbol{x},t)\otimes\boldsymbol{v}(\boldsymbol{x},t)) : \nabla_x\widetilde{\boldsymbol{w}}(\boldsymbol{x},t) \\ + \mu\nabla_x\boldsymbol{v}(\boldsymbol{x},t) : \nabla_x\widetilde{\boldsymbol{w}}(\boldsymbol{x},t)\big)\,d\boldsymbol{x}dt - \int_0^T \big\langle \boldsymbol{f}(\cdot,t), \widetilde{\boldsymbol{w}}(\cdot,t)\big\rangle_{\boldsymbol{V}^{-1},\boldsymbol{V}} dt = 0. \tag{66}$$

The regularity demands on the set of test vector-functions $\widetilde{\boldsymbol{w}}$ in this integral equality can be loosen as compared to the ones imposed on the test vector-function $\widetilde{\boldsymbol{w}}$ in (42). More certainly, relying on the standard density arguments, we can set that, in (66), $\widetilde{\boldsymbol{w}}$ merely belongs to $L^2(0,T;\boldsymbol{V})$, $\partial_t\widetilde{\boldsymbol{w}}$ merely belongs to $L^2(0,T;\boldsymbol{H})$, and $\widetilde{\boldsymbol{w}}$ vanishes in a neighborhood of the sections $\{t=0\}$ and $\{t=T\}$.

Note that the integral equality (66) is exactly the integral equality (24) with $\boldsymbol{w} = \widetilde{\boldsymbol{w}}$, which along with the inclusion $\boldsymbol{v}(\cdot,t)\in\boldsymbol{V}$ is equivalent in the sense of distributions to the system consisting of equations (21a) and (21b).

5.6. Completion of the proof of assertion 3 of Theorem 1. We divide the final part of the proof of the assertion 3 of Theorem 1 into justification of two lemmas. At first, we refine the regularity of the limit vector-function $\boldsymbol{v}$ as compared to the assertion 1 of the theorem.

Lemma 1. *Inclusions*

$$\boldsymbol{v}\in L^\infty(0,T;\boldsymbol{H})\cap L^2(0,T;\boldsymbol{V}), \quad \partial_t\boldsymbol{v}\in L^{4/d}(0,T;\boldsymbol{V}^{-1}),$$
$$\boldsymbol{v}\in C([0,T];\boldsymbol{H}) \text{ if } d=2, \quad \boldsymbol{v}\in C_{\text{weak}}([0,T];\boldsymbol{H}) \text{ if } d=3$$

hold, i.e. the limit vector-function $\boldsymbol{v}$ *meets the regularity conditions* (23).

Proof. Inclusion $\boldsymbol{v}\in L^\infty(0,T;\boldsymbol{H})\cap L^2(0,T;\boldsymbol{V})$ has already been established in the assertion 1 of the theorem.

The properties $\partial_t\boldsymbol{v}\in L^{4/d}(0,T;\boldsymbol{V}^{-1})$, $\boldsymbol{v}\in C([0,T];\boldsymbol{H})$ (in the case $d=2$) and $C_{\text{weak}}([0,T];\boldsymbol{H})$ (in the case $d=3$) appear as the classical regularity result of the theory of non-stationary Navier–Stokes equations by virtue of inclusion $\boldsymbol{v}\in L^\infty(0,T;\boldsymbol{H})\cap L^2(0,T;\boldsymbol{V})$, condition $\boldsymbol{f}\in C([0,T];\boldsymbol{V}^{-1})$, the integral equality (24) (or, equivalently, (66)), for details see, for example, [27, Ch. 3, Secs. 3.1, 3.3, 3.4]. □

At second, we show that the initial condition $\boldsymbol{v}(\cdot,0)=\overline{\boldsymbol{v}}(\cdot,1)$ holds true in the proper sense:

Lemma 2. *The limiting relations* (25) *and* (26) *hold in the respective cases* $d=2$ *and* $d=3$.

Proof. Justification of the limiting relations (25) and (26) is a natural modification of the arguments from [18, Sec. 4.5]. At first, note that, by the finite increment formula [24, Lem. 4], from (47) it follows that the family of mappings $\big(\vartheta\mapsto\overline{\boldsymbol{v}}_n(\cdot,\vartheta)\big)\colon [0,1]\mapsto\boldsymbol{V}^{-3}$ is equi-continuous. On the other hand, due to estimate (46) the values of the functions $\vartheta \mapsto \overline{\boldsymbol{v}}_n(\cdot,\vartheta)$ belong to the interval $\|\overline{\boldsymbol{v}}_n(\cdot,\vartheta)\|_{\boldsymbol{H}} \le C_0$, which is a compact subset of $\boldsymbol{V}^{-3}$. By the Arcelà–Ascoli theorem, this implies that the set $\{\overline{\boldsymbol{v}}_n\}_{n\ge n_0}$ is relatively compact in $C([0,1];\boldsymbol{V}^{-3})$. Therefore, there is a subsequence, still denoted by n, such that $\overline{\boldsymbol{v}}_n(\cdot,\vartheta) \underset{n\to+\infty}{\longrightarrow} \overline{\boldsymbol{v}}(\cdot,\vartheta)$ in $\boldsymbol{V}^{-3}$ uniformly on the segment $\{0 \le \vartheta \le 1\}$. Quite analogously, from (57) and (58) we deduce that $\widetilde{\boldsymbol{v}}_n(\cdot,\widetilde{t}) \underset{n\to+\infty}{\longrightarrow} \widetilde{\boldsymbol{v}}(\cdot,\widetilde{t})$ strongly in $\boldsymbol{V}^{-3}$ uniformly on the segment $\{0 \le \widetilde{t} \le T - 1/n_0\}$, at least for some subsequence. These two limiting relations and identities (45) and (65) imply that $\boldsymbol{v}(\cdot,0+) = \overline{\boldsymbol{v}}(\cdot,1)$ in $\boldsymbol{V}^{-3}$. Combining the latter relation with the regularity properties $\overline{\boldsymbol{v}} \in C([0,1];\boldsymbol{H})$ (for $d \in \{2,3\}$), $\boldsymbol{v} \in C([0,T];\boldsymbol{H})$ (for $d=2$), and $\boldsymbol{v} \in C_{\mathrm{weak}}([0,T];\boldsymbol{H})$ (for $d=3$), we conclude that both conditions (25) and (26) are valid.

Lemma 2 is proved. □

Thus, $\boldsymbol{v}$ is a weak solution to problem (21).

The proof of Theorem 1 is completed.

6 Concluding remarks

6.1. A note on uniqueness of the solution to system (20)–(21). Proposition 3, along with the well-known provisions of the theory of Navier–Stokes equations [27, Ch. 3, Th. 3.2], leads to the following result.

Proposition 4. *In the two-dimensional case (i.e., when $d = 2$), the "strong-weak" solution to system (20)–(21) in the sense of Definitions 2 and 3 is unique.*

6.2. A note on the strong solution to problem (21). Since problem (20) is linear, assuming that the data Ω, $\boldsymbol{v}_0$, Φ, $\boldsymbol{b}$, $\mathbb{B}$ are sufficiently smooth, we can obtain as much regularity as desired for $\overline{\boldsymbol{v}}$ and $\overline{p}_*$. In particular, the following result is valid.

Lemma 3. *In addition to the conditions imposed on the given data in Section 1, assume that $\boldsymbol{v}_0 \in \boldsymbol{V}^2$, $\boldsymbol{b} \in C^2(\overline{\Omega})^d$, $\mathbb{B} \in C^2(\overline{\Omega})^{d\times d}$, and $\Phi \in C_0^2([0,1])$. Then the strong solution $\overline{\boldsymbol{v}}$ of problem (20) has the additional regularity*

$$\overline{\boldsymbol{v}} \in C([0,1];\boldsymbol{V}^2), \tag{67a}$$

$$\partial_\vartheta \overline{\boldsymbol{v}} \in L^\infty(0,1;\boldsymbol{V}^2). \tag{67b}$$

Proof. Justification of this lemma is quite standard in the theory of evolutionary equations and simulates the proofs of Proposition 1.2 from [27, Ch. 3] and Lemma 2 from [3]. Therefore, we present it here rather schematically.

At first, from Proposition 3 we recall that the strong solution $\overline{\boldsymbol{v}}$ is unique and note that it can be constructed by the Galerkin method as the limit of the sequence of smooth Galerkin's approximations $\overline{\boldsymbol{v}}_m$ ($m \in \mathbb{N}$), as $m \to +\infty$.

At second, by virtue of the Galerkin system, we construct the set of three energy estimates

$$\|\overline{\boldsymbol{v}}(\cdot,\vartheta)\|_{\boldsymbol{H}} \le M_4\|\boldsymbol{v}_0\|^2_{\boldsymbol{H}} + M_5, \quad \forall\,\vartheta\in[0,1], \tag{68}$$

$$\|\nabla_x\overline{\boldsymbol{v}}(\cdot,\vartheta)\|^2_{L^2(\Omega)^{d\times d}} \le M_6\|\nabla_x\boldsymbol{v}_0\|^2_{L^2(\Omega)^{d\times d}} + M_7, \quad \forall\,\vartheta\in[0,1], \tag{69}$$

$$\|\Delta_x\overline{\boldsymbol{v}}(\cdot,\vartheta)\|^2_{\boldsymbol{H}} \le M_8\|\Delta_x\boldsymbol{v}_0\|^2_{\boldsymbol{H}} + M_9, \quad \forall\,\vartheta\in[0,1], \tag{70}$$

where the constants M_4–M_9 depend only on d, Ω, $\|\boldsymbol{b}\|_{C^2(\overline{\Omega})^d}$, and $\|\mathbb{B}\|_{C^2(\overline{\Omega})^{d\times d}}$.

From (68)–(70), and the well-known inequality [19, Ch. 1, Sec. 1.5, formula (16)]

$$\|\boldsymbol{\phi}\|_{W^{2,2}(\Omega)^d} \le C_\star(\Omega)\|\Delta_x\boldsymbol{\phi}\|_{L^2(\Omega)^d}, \quad \forall\,\boldsymbol{\phi}\in W^{2,2}(\Omega)^d\cap W^{1,2}_0(\Omega)^d,$$

it follows that $\overline{\boldsymbol{v}}\in L^\infty(0,1;\boldsymbol{V}^2)$, which along with the conditions $\boldsymbol{b}\in C^2(\overline{\Omega})^d$, $\mathbb{B}\in C^2(\overline{\Omega})^{d\times d}$ and $\Phi\in C^2_0([0,1])$ gives that

$$\partial_\vartheta\overline{\boldsymbol{v}} = \Phi\mathcal{P}_{\mathrm{sol}}(\boldsymbol{b}+\mathbb{B}\overline{\boldsymbol{v}})\in L^\infty(0,1;\boldsymbol{V}^2),$$

i.e. inclusion (67b) holds true.

Finally, inclusion (67a) follows from (67b) by [24, Lem. 4].

Lemma 3 is proved. □

By Lemma 3, we immediately conclude that, if $\boldsymbol{v}_0\in\boldsymbol{V}^2$, $\boldsymbol{b}\in C^2(\overline{\Omega})^d$, $\mathbb{B}\in C^2(\overline{\Omega})^{d\times d}$, and $\Phi\in C^2_0([0,1])$, then

$$\overline{\boldsymbol{v}}(\cdot,1)\in\boldsymbol{V}^2. \tag{71}$$

Thus, Lemma 3 and the well-known provisions of the theory of Navier–Stokes equations [27, Ch. 3, Th. 3.5, 3.6] lead to the following result in the two-dimensional case.

Theorem 2. *Assume that $d=2$, $\partial\Omega\in C^3$, $\boldsymbol{v}_0\in\boldsymbol{V}^2$, $\boldsymbol{f}\in C([0,T];\boldsymbol{H})$, $\partial_t\boldsymbol{f}\in C([0,T];\boldsymbol{V}^{-1})$, $\boldsymbol{b}\in C^2(\overline{\Omega})^d$, $\mathbb{B}\in C^2(\overline{\Omega})^{d\times d}$, and $\Phi\in C^2_0([0,1])$.*

Then the pair of solutions $\overline{\boldsymbol{v}}$ and $\boldsymbol{v}$ to the respective problems (20) *and* (21) *has the additional regularity*

$$\overline{\boldsymbol{v}}\in C([0,1];\boldsymbol{V}^2), \quad \partial_\vartheta\overline{\boldsymbol{v}}\in L^\infty(0,1;\boldsymbol{V}^2), \tag{72a}$$

$$\boldsymbol{v}\in L^\infty(0,T;\boldsymbol{V}^1\cap W^{2,2}(\Omega)^2), \quad \partial_t\boldsymbol{v}\in L^2(0,T;\boldsymbol{V})\cap L^\infty(0,T;\boldsymbol{H}). \tag{72b}$$

Remark 1. *The presence of the regularity properties* (72b) *means that the weak solution to problem* (21)*, under the assumptions of Theorem 2, actually becomes the strong solution in the usual sense. (The exact notion of strong solution is given, for example, in* [8, Ch. V, Sec. 2]*.)*

6.3. Two examples of explicit formulas for the "corrected" initial velocity $\overline{\boldsymbol{v}}(\cdot,1)$. Equation (20a) has a rather simple form and therefore can be easily integrated explicitly for some fairly simple matrices $\mathbb{B}$, which leads to explicit formulas of the "corrected" initial velocity field $\overline{\boldsymbol{v}}(\cdot,1)$. In conclusion of this article, let us observe two results in this direction in the two-dimensional case and draw a few notable conclusions.

Firstly, assume that

$$\boldsymbol{b} = 0, \quad \mathbb{B} = \begin{pmatrix} a\cos\phi & a\sin\phi \\ -a\sin\phi & a\cos\phi \end{pmatrix}, \tag{73}$$

where values $a \in \mathbb{R}$ and $\phi \in [0, 2\pi)$ are constant and given. For simplicity and clarity of representation of solutions, also assume that

$$\Omega \subset \{\boldsymbol{x} \in \mathbb{R}^2 \colon\ x_1 > 0,\, x_2 > 0\} \tag{74}$$

and that the given initial velocity field $\boldsymbol{v}_0$ belongs to $\boldsymbol{V}$ and is extended beyond Ω by zero. Denote

$$\Psi(\vartheta) \overset{\text{def}}{=} \exp\left\{a\cos\phi \int_0^{\vartheta} \Phi(\vartheta')d\vartheta'\right\}.$$

With these given data and with account of the above notation, the unique strong solution of problem (20) has the form

$$\overline{\boldsymbol{v}}(\boldsymbol{x}, \vartheta) = \Psi(\vartheta)\boldsymbol{v}_0(\boldsymbol{x}), \quad (\boldsymbol{x}, \vartheta) \in \overline{\Omega} \times [0, 1], \tag{75}$$

and the associated pressure function is defined by the formula

$$\overline{p}_*(\boldsymbol{x}, \vartheta) = (a\sin\phi)\Phi(\vartheta)\Psi(\vartheta)\int_0^{x_1} v_{02}(x_1', x_2)dx_1' + \text{const}, \ \ (\boldsymbol{x}, \vartheta) \in \overline{\Omega} \times [0, 1]. \tag{76}$$

The validity of representations (75) and (76) for the solution is established by their direct substitution into equations (20a), (20b) and subsequent calculation with additional account of condition $\operatorname{div}_x \boldsymbol{v}_0 = 0$.

By virtue of (4) and (75), we have

$$\overline{\boldsymbol{v}}(\boldsymbol{x}, 1) = e^{a\cos\phi}\boldsymbol{v}_0(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Omega. \tag{77}$$

Now, we formulate a simple result that follows from Theorem 1, Proposition 4 and formula (77):

Theorem 3. *Assume $d = 2$ and conditions (2)–(7), (73), (74), and $\boldsymbol{v}_0 \in \boldsymbol{V}$ hold; then $\boldsymbol{v} = \lim\limits_{n\to+\infty} \boldsymbol{v}_n$ is the unique weak solution of the initial-boundary value problem for the system of Navier–Stokes equations (21a), (21b) endowed with the no-slip boundary condition (21d) and the "corrected" initial condition*

$$\boldsymbol{v}(\boldsymbol{x}, 0+) = e^{a\cos\phi}\boldsymbol{v}_0(\boldsymbol{x}) \quad \textit{for } \boldsymbol{x} \in \Omega. \tag{78}$$

Remark 2. *If $\mathbb{B}$ is given by (73) and $\boldsymbol{v}_0 \in \boldsymbol{V}$ then one has*

$$\mathbb{B}\boldsymbol{v}_0 = (a\cos\phi)\boldsymbol{v}_0 + (a\sin\phi)\boldsymbol{v}_0^{\perp}, \quad \textit{where} \ \ \boldsymbol{v}_0^{\perp} = (v_{02}, -v_{01});$$

hence, by the Weyl decomposition, $\mathcal{P}_{\text{sol}}(\mathbb{B}\boldsymbol{v}_0) = (a\cos\phi)\boldsymbol{v}_0$ and condition (9) takes the form

$$\boldsymbol{v}(\boldsymbol{x}, 0+) = (1 + a\cos\phi)\boldsymbol{v}_0(\boldsymbol{x}).$$

Note that this condition does not coincide with condition (78), except for the cases, when $a = 0$, or $\phi = \pi/2$, or $\phi = 3\pi/2$. Thus, we confirm a conjecture that was announced in Section 1: in general, the system consisting *of equations* (8)*,* (1b)*,* (1d)*, and* (9)*, is not a proper approximation of* (1) *for large* $n \in \mathbb{N}$*.*

Secondly, assume that

$$\boldsymbol{b} \text{ satisfies conditions (2)}, \quad \mathbb{B} = 0, \tag{79}$$

and Φ and $\boldsymbol{v}_0$ satisfy conditions (4) and (7), respectively.

By the Weyl decomposition, one has

$$\boldsymbol{b} = \boldsymbol{b}_{\rm sol} + \nabla_x \overline{\pi}_*, \quad \boldsymbol{b}_{\rm sol} = \mathcal{P}_{\rm sol}(\boldsymbol{b}) \in \boldsymbol{H}, \quad \overline{\pi}_* \in W^{1,2}(\Omega), \tag{80}$$

where $\boldsymbol{b}_{\rm sol}$ and $\overline{\pi}_*$ are considered uniquely given, because $\boldsymbol{b}$ is given.

With these given data, the unique strong solution to problem (20) has the form

$$\overline{\boldsymbol{v}}(\boldsymbol{x}, \vartheta) = \boldsymbol{v}_0(\boldsymbol{x}) + \boldsymbol{b}_{\rm sol}(\boldsymbol{x}) \int_0^{\vartheta} \Phi(\vartheta')d\vartheta', \quad (\boldsymbol{x}, \vartheta) \in \overline{\Omega} \times [0,1], \tag{81}$$

and the associated pressure function is defined by the formula

$$\overline{p}_*(\boldsymbol{x}, \vartheta) = \Phi(\vartheta)\overline{\pi}_*(\boldsymbol{x}) + \text{const}, \quad (\boldsymbol{x}, \vartheta) \in \overline{\Omega} \times [0,1]. \tag{82}$$

The validity of representations (81) and (82) for the solution is established by their direct substitution into equations (20a), (20b) and subsequent calculation.

By virtue of (4) and (81), we have

$$\overline{\boldsymbol{v}}(\boldsymbol{x}, 1) = \boldsymbol{v}_0(\boldsymbol{x}) + \boldsymbol{b}_{\rm sol}(\boldsymbol{x}) \overset{(80)_2}{=} \boldsymbol{v}_0(\boldsymbol{x}) + \mathcal{P}_{\rm sol}(\boldsymbol{b})(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \Omega. \tag{83}$$

Now, we formulate a simple result that follows from Theorem 1, Proposition 4 and formula (83):

Theorem 4. *Assume* $d = 2$*, conditions* (2)*–*(7) *and* $\mathbb{B} = 0$ *hold.*

Then $\boldsymbol{v} = \lim\limits_{n\to+\infty} \boldsymbol{v}_n$ *is the unique weak solution to the initial-boundary value problem for the system of Navier–Stokes equations* (21a)*,* (21b) *endowed with the no-slip boundary condition* (21d) *and the "corrected" initial condition*

$$\boldsymbol{v}(\boldsymbol{x}, 0+) = \boldsymbol{v}_0(\boldsymbol{x}) + \mathcal{P}_{\rm sol}(\boldsymbol{b})(\boldsymbol{x}) \quad \textit{for } \boldsymbol{x} \in \Omega. \tag{84}$$

Remark 3. *Condition* (84) *is exactly condition* (9) *with* $\mathbb{B} = 0$*. Thus we conclude that, if the impulsive term in* (1a) *does not depend on solution* $\boldsymbol{v}_n$*, then the singular term* $\boldsymbol{F} = \delta_{(t=0)}\boldsymbol{b}(\boldsymbol{x})$ *is its proper approximation for large* $n \in \mathbb{N}$*. As it has already been noticed in Section 1, the result of Theorem 4 is in good agreement with the classical formulations of problems on exterior instantaneous force action on hydrodynamic flows* [6, 20, 22, 28].

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