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A BRIEF PROOF OF THE EXISTENCE OF AN α-CORE FOR GENERALIZED STRATEGY GAMES

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Abstract: This paper studies the problem of whether an α-core exists (is non-empty) for generalized normal-form games where player payoffs are described by point-to-set convex-valued, irreflexive, and weakly continuous preferences. We propose a short effective proof of the non-emptiness of the core; it based on a contradiction with Brouwer's theorem on the existence of a fixed point for a continuous mapping of a convex compact set into itself.

Keywords: generalized strategic game, α-core, Brouwer's fixed point theorem.

1 Introduction

In game theory and its applications, the concept of core is one of the key notions of the "solution" for any type of game model: in cooperative games specified via a characteristic function and also in a strategic (normal) form. The core includes such admissible outcomes (profile of individual strategies)

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that are not blocked by (admissible) coalitions of players—taking into account possible opposition from the members of a complementary coalition. Initially, the study of the core concept focused on games presented in characteristic function form—a mapping assigning winning options to (admissible) coalitions: summed payoffs (transferable utility or games with side payments) or individualized (non-transferable) payoffs. In [3, 4], Bondareva introduced balancedness, a crucial condition for the existence of core payoffs, and Shapley [12] later discussed it. Further, Scarf fruitfully extended this notion to NTU-games in [10]; one can see also [9], [7].

In [1] Aumann first proposed the concepts of the α-core and β-core for strategic games, with the former subsequently receiving extensive analysis in the literature. Essentially, if a coalition attempts to alter the game's current social state, a complementary coalition can counter this by adopting a strategy that harms at least one member of the original coalition, thereby preventing the change because that member would lose, and this is a way for discouraging the change. Thus, the counter-coalition destroys the cooperation of the opponents and counters the threat of changing the outcome of the game. The stability of outcomes from the β-core is much stronger—when repelling an attempt to change the outcome, the complementary coalition must have a counter-strategy that destroys the aggressor's cooperation, common to all kinds of change options. Accordingly, the β-core is a subset of the α-core and has (much) less chance of existence.

Scarf [11] first proved the existence of α-core solutions for n-person normal-form games having quasi-concave payoff functions and a convex compact set of possible outcomes (strategy profiles). In [6] Kajii generalized Scarf's existence theorem to encompass n-person normal-form games with general preferences. Moreover, the description of the strategy sets in these games was as convex compact subsets of a potentially infinite-dimensional *normed* vector space.

And this is the main drawback of Kajii's work—unlike finite-dimensional spaces, where one characterizes a compact set as norm-bounded and closed, infinite-dimensional normed spaces lack such a simple characterization[1]. However, Kajii's proof relies heavily on compactness, otherwise the proof method used is ineffective. One can implement the latter using weak topologies and Alaoglu's theorem, where, in exactly the same way as in a finite-dimensional space, closed and bounded sets in the original normed space are weakly star-compact sets.

Paper [8] overcames this shortcoming—the normed strategy space—and a number of other weaknesses by using Bewley's method. This involved first proving a finite-dimensional result and then performing the limit passage

[1]For example, one can recall the Arzela-Ascoli theorem, which characterizes compact sets in spaces of type $C(X)$, where $X \subset \mathbb{R}$ is a compact subset.

over a network of finite-dimensional subspaces to transition to an infinite-dimensional Hausdorff space. Furthermore, [8] clarifies the concept of the α-core itself. In the context of a generalized game, they weaken the requirements for coalition dominance by limiting the ability of a complementary coalition to influence blocking outcomes. The existence of an element of the α-core itself in the finite-dimensional setting is proved using the construction proposed in [5] in the context of the Arrow-Debreu model of the economy, supplemented by one essential element. In general, the question reduces to Kakutani's theorem on the existence of a fixed point for the constructed (a cumbersome and nontrivial) point-to-set mapping. In literature there are other studies of α-core in close to our context, for example [2] (the study of discontinuous games), also see [13] for infinite-player generalizations of Scarf's theorem and so on.

The presented paper follows the setup and assumptions described in [8], within which we give a new original proof of the non-emptiness of the α-core, short and clearly demonstrating the significance of the assumptions made. Our idea of the proof is to get a contradiction with Brouwer's theorem on the existence of a fixed point for a continuous mapping of a convex compact set into itself. In the next section we describe mathematical settings, after that we formulate and prove the main result—the existence of α-core. Presentation of an example illustrating our method of proof and conclusion remarks finish the paper.

2 Generalized Strategic Games

We consider a cooperative game environment in the context of a generalized normal-form game with a finite set of players (agents) $I = \{1, 2, \dots, n\}$. Here, any nonempty subset $S \subseteq I$ is called a coalition of players; further, we assume that in the game there are defined a set of admissible coalitions $\mathbb{I}$, such that I and each single-element coalition $\{i\}$ belong to $\mathbb{I}$. Agent $i \in I$ chooses an individual strategy $x_i \in X_i$ within its own set of admissible strategies X_i.

Assumption A1. *For each agent $i \in I$, the set of strategies X_i is a nonempty and convex subset of the finite-dimensional topological vector space L.*

We call an element $x = (x_i)_{i \in I} \in X = \prod_{i \in I} X_i$ a joint strategy (or strategic profile), and we consider it as a social state. If $S \subseteq I$, then X^S denotes the Cartesian product $\prod_{i \in S} X_i$. For simplicity, we denote the set $I \setminus S = -S$ in the standard way. We denote elements of joint strategies of coalition $S \subseteq I$ as $y^S \in X^S$.

Assumption A2. *For each feasible coalition $S \in \mathbb{I}$, a nonempty compact set $F^S \subseteq X^S$ of feasible joint strategies exists; furthermore, the set $F = F^I$ is convex.*

We consider the case where agents have (possibly unordered) preferences, in the presence of externalities (i.e., preference interdependence). Formally, each player i has a preference relation on X, which is described by a correspondence $\mathcal{P}_i$ from X to X. For any outcome $x \in X$, the set $\mathcal{P}_i(x)$ is the set of allocations $y \in X$ that are strictly preferred by player i to profile x. We make the standard assumption of convex irreflexivity of preferences.

Assumption A3. *For each $x \in X$ and $i \in I$, $x \notin \operatorname{co} \mathcal{P}_i(x)$ holds.*

Thus, the generalized game under study is represented by the following set of parameters

$$G = \langle L, (X_i, \mathcal{P}_i)_{i \in I}, F, (F^S)_{S \in \mathbb{I}} \rangle.$$

Let F_i be the set of all individual strategies $z_i \in X_i$ of player i that can be applied in $z = (z_j) \in F^T$, where $T \in \mathbb{I}(i)$ is a feasible coalition containing i and distinct from the coalition of all players I, i.e.

$$F_i = \{\pi_i(z) \mid z \in F^T, i \in T \in \mathbb{I}, T \neq I\}.$$

Here $\pi_i(z)$ denotes a natural projection of $z = (z_j)_{j \in T} \in F^T$ into X_i, where $\pi_i(z) = z_i$. Thus, the set F_i includes all strategies that person i can use when joining any coalition other than a complete one.

Definition 1. *A coalition $S \in \mathbb{I}$ α-**blocks** a strategic profile $x \in F$ if S has a joint feasible strategy $y^S \in F^S$ such that each member of S strictly prefers the outcome (state) (y^S, w^{-S}) to the outcome x, regardless of the reaction $w^{-S} = (w_j)_{j \notin S}$ of the complementary coalition, where each w_j belongs to the set $W_j = \operatorname{co} F_j$.*

*The bundles $x \in F$ such that there is no coalition capable of α-blocking x forms the α-**core** of the game G.*

The above definition can now be reformulated in the following way: If S is a feasible coalition, then let P^S be a correspondence from X to X^S, defined as

$$P^S(x) = \{y^S \in X^S \mid y^S \times W^{-S} \subset \bigcap_{i \in S} \mathcal{P}_i(x)\},$$

where $y^S \times W^{-S} = \{(y^S, w) \mid w \in W^{-S}\}$. Following this notation, a feasible joint strategy $x \in F$ belongs to the α-core if and only if there is no feasible coalition $S \in \mathbb{I}$ such that $F^S \cap P^S(x) \neq \emptyset$. Note that for every feasible coalition $S \in \mathbb{I}$ we have

$$F^{-S} \subseteq W^{-S} \subseteq X^{-S}.$$

Now we recall the notion of a "balanced" n-person game. It first appears in [3, 4] and [12] later discussed it in the context of a game with transferable utility (a game with side-payments). Scarf applied the notion of balancedness to a game in characteristic function form in [10] and in [11] he applied it to a game in normal form.

Let Δ be the set of scalar bundles $\lambda = (\lambda_S)_{S\in\mathbb{I}} \in \mathbb{R}^{\mathbb{I}}_{+}$ satisfying the condition

$$\forall i \in I, \quad \sum_{S\in\mathbb{I}, i\in S} \lambda_S = 1 \quad \Longleftrightarrow \quad \sum_{S\in\mathbb{I}} \lambda_S \mathbf{1}_S = \mathbf{1}_I,$$

where $\mathbf{1}_S(i) = 1$ for $i \in S$ and $\mathbf{1}_S(i) = 0$ otherwise. The element $\lambda \in \Delta$ is called a balanced covering (or weights), and the family of coalitions $\{S \in \mathbb{I} \mid \lambda_S > 0\}$ associated with it is called balanced.

Definition 2. *A generalized normal form game*

$$G = \langle I, \{X_i, \mathcal{P}_i(\cdot)\}_{i\in I}, F, (F^S)_{S\in\mathbb{I}}\rangle$$

is called ***balanced*** *if for every balanced covering $\lambda \in \Delta$, and every set of feasible strategies $y^S \in F^S$ with $S \in \mathbb{I}$ and $\lambda_S > 0$, the state $z = (z_i)_{i\in I}$ defined as*

$$\forall i \in I \ z_i = \sum_{S\in\mathbb{I}: i\in S} \lambda_S y_i^S, \tag{1}$$

is ***feasible****, i.e., $z \in F$.*

Finally, we will also assume a specific type of continuity of preferences.

Definition 3. *A generalized game G is called α-continuous if for every feasible coalition $S \in \mathbb{I}$ the correspondence P^S has open lower sections in X, that is, for every $i \in S$ and every feasible strategy $y^S \in F^S$, the set $\{x \in X \mid y^S \times W^{-S} \subset \mathcal{P}_i(x)\}$ is open in X.*

Notice this is true if $\mathcal{P}_i(\cdot)$ has open graph for every $i \in I$. So, the assumption is reasonably unrestrictive.

3 Theorem on Non-emptiness of α-core

Theorem 1. *Under assumptions A1-A3 if the game*

$$G = \langle I, \{X_i, \mathcal{P}_i(\cdot)\}_{i\in I}, F, (F^S)_{S\in\mathbb{I}}\rangle$$

is α-continuous and balanced, then α-core is nonempty, i.e. $\mathcal{C}_\alpha(G) \neq \emptyset$.

Proof. By contradiction. Assuming $\mathcal{C}_\alpha(G) = \emptyset$ we show that does exist a continuous mapping $f : F \to F$ that has no fixed point, which contradicts Brouwer's theorem. We describe the construction of this mapping below.

The assumption $\mathcal{C}_\alpha(G) = \emptyset$ means that for each $x = (x_i)_{i\in I} \in F$ there exists a coalition $S \in \mathbb{I}$ that blocks x, i.e.

$$\exists y^S \in \mathcal{P}^S(x) \cap F^S \neq \emptyset.$$

Moreover, by the assumption of α-continuity of G, for some $V(x) \subset X$ open in X, one has

$$x \in V(x) \ \ \& \ \ \forall x' \in V(x) \ \ y^S \in \mathcal{P}^S(x') \cap F^S.$$

Here $V(x)$ is an open neighborhood of $x \in F$. Since $V(x)$ form a covering of the compact set F, there is a finite subcovering $V_k = V(x^k)$, and the corresponding coalition $S_k \subseteq I$, α-blocking every $x \in V_k$, $k = 1, 2, \ldots, m$.

Next, we consider the continuous decomposition of the identity corresponding to the family V_k, i.e., we find a set of continuous functions $\eta_k : F \to [0,1]$ such that

$$\forall x \in F \ \sum_{k=1}^{m} \eta_k(x) = 1 \ \ \& \ \ \eta_k(x) > 0 \iff x \in V_k, \quad k = 1, 2, \dots, m.$$

Now, for an arbitrary chosen and fixed $z = (z^{S_k}, z^{-S_k}) \in \prod_{i \in I} F^{\{i\}}$, we define on F the function $h(\cdot)$ by the formula

$$h(x) = \sum_{k=1}^{m} \eta_k(x)(y^{S_k}, z^{-S_k}).$$

The continuity of $h(\cdot)$ follows from the construction. Next, we specify

$$\theta(x) = \min_{i \in I} \sum_{k: i \notin S_k} \eta_k(x) < 1$$

and consider

$$f(x) = \frac{h(x) - \theta(x) z}{1 - \theta(x)}.$$

The function $f(x) = (f_1(x), \dots, f_n(x))$ is the desired one; we will show that it has the required properties. One has

$$\forall i \in I \quad f_i(x) = \sum_{k: i \in S_k} \lambda_k(x) y_i^{S_k} + \lambda_{m+i}(x) z_i,$$

where

$$\lambda_k(x) = \frac{\eta_k(x)}{1 - \theta(x)} \quad \& \quad \lambda_{m+i}(x) = \frac{\sum_{k: i \notin S_k} \eta_k(x) - \theta(x)}{1 - \theta(x)}.$$

By construction, the set $\lambda = \{\lambda_\xi\}_{\xi=1}^{m+n}$ forms a balanced set of weights for the family of coalition strategies $y^{S_k} \in F^{S_k}$, $k = 1, \dots, m$ and $z_i \in F^{\{i\}}$, $i = 1, \dots, n$. The assumption of a balanced game implies $f(x) \in F$.

Moreover, there exists an individual $i \in I$ such that $\theta(x) = \sum_{k: i \notin S_k} \eta_k(x)$ $\Rightarrow \lambda_{m+i}(x) = 0$. Therefore, we can write the components of $f(x)$ as

$$f_i(x) = \sum_{k: i \in S_k} \lambda_k(x) y_i^{S_k},$$

and for other $j \in I$, $j \neq i$ we have

$$f_j(x) = \sum_{k: i, j \in S_k} \lambda_k(x) y_j^{S_k} + \sum_{k: i \in S_k, j \notin S_k} \lambda_k(x) w_j(x),$$

where

$$w_j = \frac{\sum_{k: j \in S_k, i \notin S_k} \lambda_k(x) y_j^{S_k} + \lambda_{m+j}(x) z_j}{\sum_{k: j \in S_k, i \notin S_k} \lambda_k(x) + \lambda_{m+j}(x)} \in F_j.$$

Since $\sum_{k:j\in S_k, i\notin S_k} \lambda_k(x) + \lambda_{m+j}(x) = \sum_{k:j\notin S_k, i\in S_k} \lambda_k(x)$, then $f(x)$ is represented as a convex combination of vectors with numbers $k : i \in S_k$ such that for $w^{-S_k} = (w_j)_{j\notin S_k} \in W^{-S_k}$ we have

$$(y^{S_k}, w^{-S_k}) \in \bigcap_{j\in S_k} \mathcal{P}_j(x) \subseteq \mathcal{P}_i(x).$$

Therefore, $f(x) \in \operatorname{co}\mathcal{P}_i(x)$. However, now if we assume that there is a fixed point $\bar{x} = f(\bar{x})$, then for this player, we get a contradiction with the assumed in A3 convex irreflexivity: $\bar{x} \notin \operatorname{co}\mathcal{P}_i(\bar{x})$, which is what we had to prove. □

The following example illustrates the principle underlying the construction of the function $f(x)$ that yields the desired contradiction.

Example 1. Consider a three-player game, and let $x = (x_1, x_2, x_3) \in F$ be an outcome that is blocked by coalitions $S_1 = \{1, 2\}$, $S_2 = \{1, 3\}$, and $S_3 = \{2, 3\}$. Suppose $\eta_1(x) = 1/2$, $\eta_2(x) = 1/3$ and $\eta_3(x) = 1/6$ are the coefficients of the unity decomposition, defining

$$h(x) = \frac{1}{2}(y_1^{S_1}, y_2^{S_1}, z_3) + \frac{1}{3}(y_1^{S_2}, z_2, y_3^{S_2}) + \frac{1}{6}(z_1, y_2^{S_3}, y_3^{S_3}) =$$

$$\left(\frac{1}{2}y_1^{S_1} + \frac{1}{3}y_1^{S_2} + \frac{1}{6}z_1, \frac{1}{2}y_2^{S_1} + \frac{1}{6}y_2^{S_3} + \frac{1}{3}z_2, \frac{1}{3}y_3^{S_2} + \frac{1}{6}y_3^{S_3} + \frac{1}{2}z_3\right).$$

Here $z = (z_1, z_2, z_3) \in F^{\{1\}} \times F^{\{2\}} \times F^{\{3\}}$ is an arbitrarily chosen element. Next we find $\theta(x) = 1/6$ and specify

$$f(x) = \frac{h(x) - 1/6z}{1/2 + 1/3} =$$

$$= \left(\frac{3}{5}y_1^{S_1} + \frac{2}{5}y_1^{S_2}, \frac{3}{5}y_2^{S_1} + \frac{1}{5}y_2^{S_3} + \frac{1}{5}z_2, \frac{2}{5}y_3^{S_2} + \frac{1}{5}y_3^{S_3} + \frac{2}{5}z_3\right) =$$

$$\frac{3}{5}\left(y_1^{S_1}, y_2^{S_1}, \frac{1}{3}y_3^{S_3} + \frac{2}{3}z_3\right) + \frac{2}{5}\left(y_1^{S_2}, \frac{1}{2}y_2^{S_3} + \frac{1}{2}z_2, y_3^{S_2}\right).$$

Received:

$$\left(y_1^{S_1}, y_2^{S_1}, \frac{1}{3}y_3^{S_3} + \frac{2}{3}z_3\right) \in \mathcal{P}_1(x) \cap \mathcal{P}_2(x) \quad \&$$

$$\left(y_1^{S_2}, \frac{1}{2}y_2^{S_3} + \frac{1}{2}z_2, y_3^{S_2}\right) \in \mathcal{P}_1(x) \cap \mathcal{P}_3(x),$$

whence $f(x) \in \operatorname{co}\mathcal{P}_1(x)$. Moreover, $f_1(x) = \frac{3}{5}y_1^{S_1} + \frac{2}{5}y_1^{S_2}$, $f_2(x) = \frac{3}{5}y_2^{S_1} + \frac{1}{5}y_2^{S_3} + \frac{1}{5}z_2$ and $f_3(x) = \frac{2}{5}y_3^{S_2} + \frac{1}{5}y_3^{S_3} + \frac{2}{5}z_3$. It follows that the balanced covering implementing $f(x)$ is the family $\{S_1, S_2, S_3, \{2\}, \{3\}\}$, implemented by the set of weights: $\lambda_{12} = 3/5$, $\lambda_{13} = 2/5$, $\lambda_{23} = 1/5$, $\lambda_2 = 1/5$, $\lambda_3 = 2/5$.

3.1. Conclusion. The paper presents a simple short proof of an important game-theoretical concept of α-core under general assumptions, known in literature. We prove only a finite-dimensional theorem as soon as only at this part our achievements have been made. The general theorem—the case of Hausdorff linear space as a space of strategies can be proven repeating arguments of [8]. Moreover, one can hope that the method similar to applied here can be developed to effectively study existence of other core concepts; e.g. it may be β or γ cores, that still have not known good conditions under which they exist.

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