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BRANCH-AND-PRICE ALGORITHM FOR THE EFFICIENT 2-TERMINAL RELIABILITY PROBLEM

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Abstract: The Two-Terminal Reliability Problem, which assesses the probability of connection between two designated nodes in a network subject to the failures, remains a challenging and well-recognized #P-hard problem. In the recent years, along with the guaranteed reliability of the network, the question of the overall cost minimization became quite significant. In this paper, we introduce a proof-of-concept of the Branch-and-Price algorithm that relies on the well-known decomposition and column generation techniques in order to efficiently solve this problem. Numerical evaluation over benchmarking instances demonstrate the high performance of the proposed algorithm.

Keywords: 2-Terminal Reliable Problem, Erdös-Rényi random graphs, MIP models, Dantzig-Wolfe decomposition, branch-and-price.

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1 Introduction

Global conflicts, pandemics and economical instability have demonstrated how fragile and uncertain the international trade may be [2]. A recent analysis shows that annually, over 75% of businesses encounter disruptions in their supply chains resulting from diverse factors, such as labor disputes, logistical delays, severe weather events, fluctuations in currency exchange rates, energy shortages, breakdowns in service provision, and IT system failures (see e.g. [21]). Therefore, addressing the reliability and resilience of supply chains and their associated networks is a task of critical importance [8].

The most straightforward and widely known mathematical model [22, 11] designed to simulate resilient supply chains is represented by a transportation network with dedicated single supplier and customer nodes, whose reliability is described in terms of the classic Erdös-Rényi random graphs model. In this context, the problem is to minimize total operational cost, subject to reliability constraint.

In their recent paper [3], Casazza and Ceselli introduced a close optimization problem in the field of communication networks, called Resilient Path Selection Problem (RPSP). Their goal was to find an economically efficient subnet of the given network, while providing the reliability guarantees.

Evidently, provided that the network is completely reliable, the RPSP is equivalent to the classic Shortest-Path Problem and solvable to optimality by Dijkstra algorithm. On the other hand, reaching the required level of reliability is quite challenging. In the literature, the problem to verify the reliability constraint for a given network is known as the 2-Terminal Reliability Problem (2-TRP), dated to the seminal papers of Moore and Shannon [12] and Valiant [26]. Since its introduction, this problem has been studied extensively for decades by researchers in counting complexity and approximation algorithms (see e.g. [16, 6, 27, 4]).

In this paper, we introduce the Efficient 2-Terminal Reliability Problem (E2TRP) - an adaptation of both RPSP and 2-TRP for the resilient supply chains. By extending the algorithmic results of [3], we propose an exact branch-and-price algorithm for this problem and show it's ability to obtain feasible, suboptimal, and even optimal solutions by the results of comparative numerical evaluation.

2 Related Work

The E2TRP is closely related to the classic 2-Terminal Reliability Problem, included by Valiant into the list of first thirteen #P-complete problems [26]. Due to the proven intractability of such problems, many researchers have focused their efforts on design of approximation algorithms and schemes, and finding non-trivial polynomial-time solvable subclasses. In [18], authors proposed a linear-time algorithm for k-TRP over series-parallel networks. Subsequent work by [23, 24] examined two-terminal reliability using a product of transfer matrices, accounting for the independent reliability of each edge and node. Further advancements were made in [27], where a Monte Carlo-based algorithm was introduced to handle large-scale networks more efficiently. More recently, a fully polynomial-time randomized approximation scheme (FPRAS) was developed for Directed Acyclic Graphs (DAGs) in [9]. The obtained results on the 2-TRP and related problems indeed represent state-of-the-art in the field of algorithmic design. Unfortunately, they hardly applicable to an analysis of supply chain resilience, as the aspect of cost minimization is being omitted.

Another direction to ensure reliability in networks is related to design of so-called *backup paths* (see, e.g. [15, 14]). In [5], the authors approximated resilience of the supply chain in terms of the resource constrained shortest-path problem. In the similar way, Redmond et al. [17] proposed a reliable approach for multi-leg flights. Furthermore, in [13], backup facility concept was generalized for the case of more complex supply chain topology. However, as it was shown [3], the approach based on backup paths and facilities can provide solutions that may be far from optimal, or even violate the reliability constraint.

Finally, the Resilient Path Selection Problem introduced in [3], seems to be the most close to the problem considered in our paper. The authors proposed a MILP-based formulation and the respective branch-and-price algorithm for the RPSP, as well as the results of its competitive numerical evaluation. Unfortunately, their method exhibited several shortcomings, primarily induced by the non-linearity of the RPSP and intractability of the 2-TRP. In particular, in attempt to improve performance of their algorithm, the authors replaced time-consuming Moore-Shannon algorithm of reliability calculation with a Bayesian Network, which resulted in non-feasible solutions.

In this paper, we try to bridge some of these gaps. Our contribution is two-fold:

- by extending our recent results on algorithmic analysis of routing problems [19, 20] and applying the state-of-the-art achievements in column generation [7], we introduce a branch-and-price algorithm for the non-linear Efficient 2-Terminal Reliability Problem;
- numerical evaluation of the proposed algorithm against the benchmarking instances from [3] demonstrates its ability to obtain high-accuracy feasible and even optimal solutions of the studied problem.

3 Problem Statement

3.1. Informal description. It is convenient to setup the E2TTP in terms of a given transportation network G two its nodes - *supplier* and *consumer* - are dedicated and called *terminals*. Direct link providing transportation from some node u to node v is represented by the arc $e = (u, v)$, which is assigned by a *cost* (charge of use) c_e and *reliability* p_e. It is assumed that each arc can be unavailable with probability $q_e = 1 - p_e$ independently of all others. For each subnet $G' \subset G$, its cost accumulates the costs of all the constituent arcs, while the reliability is equal to the probability that terminals s and t are turn to be connected in G'.

Thus, the goal is to minimize cost of a subnet G', subject to the constraint on its reliability to be no lower than the given probability threshold.

3.2. Mathematical formulation. An arbitrary instance of the E2TRP is given by a tuple $(G, s, t, \mathfrak{F})$, where

(i) $G = (V, E, c, p)$ is an edge-weighted digraph, such that
- the node set V contains supplier s and consumer t;
- the weighting functions $c\colon E \to \mathbb{R}_+$ and $p\colon E \to [0, 1]$, to any arc $e \in E$, assign its cost c_e and reliability p_e, respectively;

(ii) the value $0 < \mathfrak{F} < 1$ is a given unreliability threshold.

We rely on the classic generalized Erdős-Rényi model (see, e.g. [1]). According to this model, the arcs of G fail mutually independently, such that each $e \in E$ turns to be unavailable (fails) with a given probability $q_e = 1 - p_e$.

A non-empty subgraph $G' = (V', E') \subset G$, $\{s, t\} \subset V'$, is called a feasible solution of E2TRP, if terminals s and t are connected in G' with probability at least $1 - \mathfrak{F}$.

The goal is to find a minimum cost feasible solution G', where

$$\mathrm{cost}(G') = \sum_{e \in E'} c_e.$$

In the sequel, we assume that, for each arc $e \in E$, $c_e > 0$ and $p_e \in (0, 1]$. We also assume that $p_e \neq 0$ because otherwise we can just remove an arc e from the graph G. Then, w.l.o.g. we can restrict ourselves to consideration of only feasible solutions that can be represented as a union of some family of possible s-t-routes $\mathfrak{R}$ in the given graph G.

4 Mixed-integer models

We start with description of a non-linear Integer Programming (IP) model of the considered problem. Following to the terminology proposed in [7], we call it an *original* formulation of our problem.

4.1. Original formulation. In our model, we encode an arbitrary feasible solution $G' \subset G$ it terms of the following decision variables:

- $\xi_e \in \{0, 1\}$, where $\xi_e = 1$, if $e \in E'$;
- $x_0^\pi \in \{0, 1\}$, where $x_0^\pi = 1$, if the route π belongs to G';
- $x_e^\pi \in \{0, 1\}$, where $x_e^\pi = 1$, if the arc e belongs to π.

In addition, we employ the following technical notation: $\mathbf{x}^\pi = [x_0^\pi, x_e^\pi : e \in E]$. Thus, we obtain the following integer program:

$$\text{(IP)} : \min \sum_{e \in E} c_e \xi_e \tag{1}$$

$$s.t.$$

$$\mathbb{P}(G') \leq \mathfrak{F} \tag{2}$$

$$x_e^\pi \leq \xi_e, \ (e \in E), \qquad (\pi \in \mathfrak{R}) \tag{3}$$

$$x_e^\pi \leq x_0^\pi, \qquad (e \in E) \tag{4}$$

$$\sum_{(i,j) \in E} x_{(i,j)}^\pi - \sum_{(j,i) \in E} x_{(j,i)}^\pi = \begin{cases} x_0^\pi, & i = s \\ -x_0^\pi, & i = t \\ 0, & \text{otherwise} \end{cases} \qquad (\pi \in \mathfrak{R}) \tag{5}$$

$$\mathbf{x}^\pi \neq \mathbf{x}^\sigma, \qquad (\pi, \sigma \in \mathfrak{R}) \tag{6}$$

$$x_e^\pi, x_0^\pi, \xi_e \in \{0, 1\}, \quad \begin{pmatrix} e \in E \\ \pi \in \mathfrak{R} \end{pmatrix}. \tag{7}$$

The objective (1) is to minimize the cost of the subgraph G', while equation (2) defines a reliability constraint. Equations (3) and (4) establish relations between corresponding decision variables. Equations (5) set the flow conservation constraints, while equations (6) guarantee uniqueness of routes that induce subgraph G'.

4.1.1. Probability Relaxation. Generally speaking, the model IP is nonlinear and makes difficult application of the traditional column generation approach. Therefore, we proceed with the relaxation of this model relying on the following technical statement.

Theorem 1 ([3])**.** *Let $\pi_1, \pi_2, \ldots, \pi_k$ be the routes in G, where route π_i fails with probability $\mathbb{P}(\pi_i)$, while all of them fail simultaneously with probability $\mathbb{P}(\pi_1, \pi_2, \ldots, \pi_k)$. Then, $\prod_{i=1}^k \mathbb{P}(\pi_i) \leq \mathbb{P}(\pi_1, \pi_2, \ldots, \pi_k)$.*

Suppose that subgraph G' is induced by some routes $\pi_1, \pi_2, \ldots, \pi_k$. By Theorem 1, $\prod_{i=1}^k \mathbb{P}(\pi_i) \leq \mathbb{P}(G')$. Therefore, inequality (2) implies $\prod_{\pi \in \mathfrak{R}} \mathbb{P}(\pi)^{x_0^\pi} \leq \mathfrak{F}$, i.e.

$$\sum_{\pi \in \mathfrak{R}} \tilde{\mathbb{P}}(\pi) x_0^\pi \leq \tilde{\mathfrak{F}}, \tag{8}$$

where $\tilde{\mathbb{P}}(\pi) = \log_2 \mathbb{P}(\pi)$ and $\tilde{\mathfrak{F}} = \log_2 \mathfrak{F}$. In the sequel, along with IP model, we consider its relaxation RP, obtained by replacing (2) with (8).

4.2. Dantzig-Wolfe decomposition. Since both IP and RP are non-compact formulations, we apply the column generation based methods [7, 25]. As it follows from the structure of those formulations, it is convenient to employ the classic Dantzig-Wolfe decomposition. Indeed, it is easy to see that for an arbitrary route $\pi \in \mathfrak{R}$, $\mathbf{x}^\pi \in \mathcal{D}$, where

$$\mathcal{D} = \left\{ \mathbf{x} \in \{0,1\}^{|E|+1} \;\middle|\; \begin{array}{lr} \sum\limits_{(i,j)\in E} x_{(i,j)} - \sum\limits_{(j,i)\in E} x_{(j,i)} = & \begin{cases} x_0, \; i = s \\ -x_0, \; i = t \\ 0, \; \text{otherwise} \end{cases} \\ x_e \le x_0, & (e \in E) \end{array} \right\}.$$

By Minkowski-Weyl theorem, an arbitrary vector $\mathbf{x} \in \operatorname{conv}(\mathcal{D})$ is represented as a convex combination of the extreme points $p \in P$ of this polytope:

$$\mathbf{x} = \sum_{p\in P} \mathbf{x}^p \lambda_p \tag{9}$$

$$\sum_{p\in P} \lambda_p = 1 \; \lambda_p \ge 0. \tag{10}$$

Furthermore, in our case, $P = \mathcal{D} = \mathfrak{R}$, and for any $\mathbf{x}^\pi \in \mathcal{D}$, $\mathbf{x}^\pi = \mathbf{x}^p$, $\lambda_p = x_0^p$, for some $p \in P$. After substitution of $\mathbf{x}^\pi$ into RP formulation, we obtain the following problem:

$$\min \sum_{e\in E} c_e \xi_e \tag{11}$$

$$s.t.$$

$$\sum_{p\in P} \tilde{\mathbb{P}}(\mathbf{x}^p)\lambda_p \le \tilde{\mathfrak{F}} \tag{12}$$

$$x_e^p \lambda_p \le \xi_e, \quad \begin{pmatrix} e \in E, \\ p \in P \end{pmatrix} \tag{13}$$

$$\lambda_p \in \{0,1\}, \qquad (p \in P) \tag{14}$$

$$\xi_e \in \{0,1\} \qquad (e \in E). \tag{15}$$

Further, we aggregate constraints (13) in order to make the obtained Master Problem smaller. In addition, in order to speed-up the subsequent branching procedure, we append the following valid inequality:

$$\sum_{p\in P} \lambda_p \ge k_{min}, \tag{16}$$

where $k_{min} = \left\lceil \frac{\tilde{\mathfrak{F}}}{\tilde{\mathbb{P}}(\mathbf{x}^{p^*})} \right\rceil$, and p^* is the most reliable s-t route in graph G.

Finally, we obtain the following Integer Master Problem:

$$(\mathrm{IMP}) : \min \sum_{e \in E} c_e \xi_e \tag{17}$$

$$s.t.$$

$$\sum_{p \in P} \tilde{\mathbb{P}}(\mathbf{x}^p) \lambda_p \leq \tilde{\mathfrak{F}} \tag{18}$$

$$\sum_{p \in P} x_e^p \lambda_p \leq M \xi_e, \quad (e \in E), \tag{19}$$

$$\sum_{p \in P} \lambda_p \geq k_{min}, \tag{20}$$

$$\lambda_p \in \{0, 1\}, \quad (p \in P) \tag{21}$$

$$\xi_e \in \{0, 1\}, \quad (e \in E) \tag{22}$$

for an appropriate big number M.

4.3. Column generation. We proceed with the LP-relaxation of model (17)-(22) that is obtained by replacing (21) and (22) by $0 \leq \lambda_p \leq 1$, $(p \in P)$ and $\xi_e \geq 0$, $(e \in E)$ respectively. To obtain its optimal solution, we apply the well-known column generation framework by considering a *Restricted Master Problem* (RMP) over some subset $P' \subset P$. To guarantee the feasibility of the RMP, we initialize it with a set of artificial variables such that each of them is penalized in the objective function. The new columns are obtained step-by-step by the following scheme.

Let α and $\beta_e, (e \in E)$ be the dual variables corresponding to the constraints (18) and (19) respectively. Then, the Dual problem of the RMP (DRMP) containing P' columns is formulated as follows:

$$(\mathrm{DRMP}(P')) : \max \ -\alpha \tilde{\mathfrak{F}} + \gamma k_{min} \tag{23}$$

$$s.t.$$

$$-\alpha \tilde{\mathbb{P}}(\mathbf{x}^p) - \sum_{e \in E} \beta_e x_e^p + \gamma \leq 0, \quad (p \in P') \tag{24}$$

$$M \beta_e - c_e \leq 0, \quad (e \in E) \tag{25}$$

$$\alpha \geq 0, \tag{26}$$

$$\beta_e \geq 0. \quad (e \in E) \tag{27}$$

Suppose that $(\bar{\alpha}, \bar{\beta}_e, \bar{\gamma})$ is an optimal solution of the DRMP(P'). In order to be appended as a new column, route p should satisfy the constraint

$$\bar{\alpha} \tilde{\mathbb{P}}(\mathbf{x}^p) + \sum_{e \in E} \bar{\beta}_e x_e^p - \bar{\gamma} < 0. \tag{28}$$

If such a route p exists, we append it to the RMP and continue the column generation process. Otherwise, the column generation procedure is complete.

In order to find such a route p:

- if $\bar{\alpha} = 0$, we find a route p satisfying $\sum_{e \in E} \bar{\beta}_e x_e^p < \bar{\gamma}$ by solving the shortest path problem in the auxiliary graph $\tilde{G} = (\tilde{V}, \tilde{E})$ where edges $e \in \tilde{E}$ have weights $\bar{\beta}_e$;
- if $\bar{\alpha} > 0$, we solve a pricing problem which is equivalent to the collection of the following auxiliary problems:

$$\mathrm{AUX}(d)\colon \min \left\{ \tilde{\mathbb{P}}(\mathbf{x}^p)\colon p \in P \setminus P', \sum_{e \in E} \bar{\beta}_e x_e^p = d \right\}. \tag{29}$$

Indeed, to find a desired route p, we have to minimize the LHS of (28) that is equivalent to minimizing $\tilde{\mathbb{P}}(\mathbf{x}^p)$ for some fixed value d.

Since $\log_2$ is a monotonically increasing function, the problem AUX(d) is equivalent to the following problem:

$$\mathrm{SUR}(d)\colon \max \left\{ \sum_{e \in E} x_e^p \log_2 p_e \colon p \in P \setminus P', \ \sum_{e \in E} \bar{\beta}_e x_e^p = d \right\}.$$

It easy to see, that the problem SUR(d) linearly depends only on x_e^p variables, which is the main advantage over the AUX(d).

In order to add a new column to the RMP, we should solve the following family of problems P(d), parametrized by values d:

$$(\mathrm{P}(d))\colon \max \sum_{e \in E} x_e \log_2 p_e \tag{30}$$

$$s.t.$$

$$\sum_{(i,j) \in E} x_{(i,j)} - \sum_{(j,i) \in E} x_{(j,i)} = \begin{cases} 1, & i = s \\ -1, & i = t \\ 0, & \text{otherwise} \end{cases} \tag{31}$$

$$\sum_{e \in E(p)} x_e \le |E(p)| - 1, \qquad (p \in P') \tag{32}$$

$$\sum_{e \in E} \bar{\beta}_e x_e = d \tag{33}$$

$$x_e \in \{0, 1\}, \qquad (e \in E). \tag{34}$$

Here, constraints (32) guarantee that no column is appended to the RMP twice, and equation (33) binds the values ξ_e with the parameter d.

If at least one of the problem P(d) provides a route satisfying (28) we add it to the RMP(P').

To reduce the number of the pricing problems, we consider the following heuristic. Since p^* is the most reliable s-t route, we obtain the following simple inequality: $\tilde{\mathbb{P}}(\mathbf{x}^{p^*}) \le \tilde{\mathbb{P}}(\mathbf{x}^p) \le -\frac{d-\gamma}{\bar{\alpha}}$, which helps us to discard problems P(d), where $d < -\bar{\alpha}\tilde{\mathbb{P}}(\mathbf{x}^{p^*}) + \gamma$.

5 Branch-and-price algorithm

We start with solution of LP-relaxation of IMP using column generation procedure discussed in Section 4. If we managed to find an integer solution satisfying the reliability constraint (2), which is validated by the application of the classic Moore-Shannon algorithm [12], then we have an optimal solution. Otherwise, we proceed with branching.

5.1. Branching scheme. Let $(\bar{\xi}_e, \bar{\lambda}_p)$ is an optimal solution of LP-relaxation of the IMP. In the case when the $\bar{\xi}_e$ values are fractional, we *branch on arcs.*

To proceed, we choose the most fractional arc $\tilde{e} = \operatorname{argmin}_{e\in E} \left|\bar{\xi}_e - 0.5\right|$ and spawn two child nodes in the branching tree. In the first child, we append constraint $\xi_{\tilde{e}} = 0$ that implies reduction of edge $\tilde{e}$ in the node subgraph. Moreover, all routes $\{p\colon x^p_{\tilde{e}} = 1\}$ are declined in the IMP. In the second child, we simplify the IMP objective and remove constraint $\sum_p x^p_{\tilde{e}}\lambda_p \le M\xi_{\tilde{e}}$. We should note that in both cases the pricing problem keeps its structure.

In the case when the obtained integer solution $\bar{\xi}_e$ violates the reliability constraint (2), we proceed with *probability branching.*. We spawn two child nodes in the branching tree as follows:

- We ruin the subgraph G' by appending the constraint $\sum_{e\in E'} \xi_e \le |E'| - 1$. In other words, we restrict to use the subgraph G' as the part of a solution.
- We save the subgraph G' as a part of solution but enhance it by an appropriate way. To append additional edges, we include the following constraints

$$\xi_e = 1,\ (e \in E'),$$
$$\sum_{e\in E} \xi_e \ge |E'| + 1.$$

Once the child nodes are obtained, we apply the column generation procedure for each of them. By construction, those pricing problems retain their structure as well.

5.2. Primal Heuristics. To speed up the convergence of the search procedure, we apply the following primal heuristics in parallel. Each time when some of them manage to find an incumbent solution, we update the record upper bound.

Start greedy heuristic. Before the column generation begins, we construct a set of independent routes as follows. At first, we find the shortest route and reduce its corresponding arcs from graph. Then, we repeat it until the graph is s-t connected or the probability condition is met. The resulted set of routes will induce a subgraph that is a new incumbent.

Local search. For any branching node we construct the following model from the routes $\{p \in P' \colon \lambda_p > 0\}$:

$$\min \sum_{e \in E} c_e \xi_e \tag{35}$$

$$\sum_{p \in P'} \tilde{\mathbb{P}}(\mathbf{x}^p)\lambda_p \leq \tilde{\mathfrak{F}} \tag{36}$$

$$\sum_{p \in P'} x_e^p \lambda_p \leq \xi_e \ (e \in E) \tag{37}$$

$$\xi_e \in \{0,1\}, \ \lambda_p \in \{0,1\} \tag{38}$$

Due to the constraints (37) and (38), the obtained solution consists of edge-disjoined s-t routes. Therefore, this solution is feasible for the initial E2TRP.

5.3. Pruning. We prune a branch any time when one of the following condition in the current node is met:

- the obtained local Lower Bound (LB) is greater than the current Upper Bound (UB) - the cost of the best known incumbent solution;
- the nodes s and t are located in different connected components of current subgraph;
- integer solution satisfying reliability constraint (2) (incumbent) is found;
- the LP-relaxation of the current MP is infeasible.

6 Numerical Evaluation

In our paper, we focus on the comparison of our algorithm with results of paper [3]. The authors of that paper did not provide neither implementation details of their algorithm, nor the benchmarking instances. Thus, we synthesize the random instances following their instructions, and compare our averaged results with theirs. For the sake of convenience, we denote the method of our colleagues as A_{CC}, and the proposed algorithm as A_{E2TRP}.

6.1. Benchmarking instances. In all the instances, graph G is specified by a finite point set V that consists of points taken at random from the square $(-100, 100) \times (-100, 100)$ on the Euclidean plane. Similarly to [3], we restrict ourselves to the case of $|V| = \{20, 30\}$, and the graph density $\gamma \in \{0.25, 0.5, 0.75\}$.

For any arc e, its cost c_e is defined by the Euclidean distance between its endpoints, and failure probability q_e taken at random from the set $\{5 \cdot 10^{-3}, 10^{-3}, 10^{-4}, 10^{-5}\}$. For any instance, we guarantee that terminals s and t belong to the same connected component.

For each tuple (V, γ), we synthesize a bunch of 10 random graphs. Then, for each generated graph we consider E2TRP instances $(V, \gamma, \mathfrak{F})$, where $\mathfrak{F} \in \{1 \cdot 10^{-1}, 5 \cdot 10^{-2}, 1 \cdot 10^{-2}, 5 \cdot 10^{-3}, 1 \cdot 10^{-3}, 5 \cdot 10^{-4}, 1 \cdot 10^{-4}, 5 \cdot 10^{-5}, 1 \cdot 10^{-5}, 5 \cdot 10^{-6}, 1 \cdot 10^{-6}\}$.

Thus, we obtain 660 instances of the E2TRP. All benchmarking instances together with source code are available by request.

6.2. Experimental setup. Our experiment consists of three subsequent stages. At the first stage, we evaluate performance of our algorithm in comparison the algorithm proposed in [3]. Since the authors of [3] report on the results obtained only for graphs of 20 nodes, we restrict ourselves to the instances of the same size. In addition, we set time limit to 1 hour (3600 sec) as well.

On the second stage, we examine our algorithm on 30 node instances to demonstrate its ability to obtain approximate solutions within the same time limit.

Finally, on the third stage, we increase the time limit to 10 hours and test the computational ability of the proposed algorithm over the whole database of the benchmarking instances and compare it with the results obtained within the hour.

The algorithm was implemented in Python 3.9.7 using NetworkX library, while the LP-relaxations of the aforementioned problems was done on top of the Gurobi MIP-solver [10].

The experiments were carried out on the 'Uran' supercomputer center at N.N. Krasovskii Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Sciences http://parallel.uran.ru, on Intel(R) Xeon(R) 16 core CPU E5-2697 v4 @2.30 GHz 256GB RAM.

6.3. Results and discussion. Results of the first stage are reported in Table 1. For each group of instances the best performers are highlighted in bold.

As it can be seen from the Table 1, the number of optimal solutions obtained by A_{CC} is greater than the one obtained by A_{E2TRP} but our algorithm provides branching trees of less nodes in average and less average relative errors (gaps) for the instances of tight probability thresholds $\mathfrak{F}$. For instance, for the group $(20, 0.5, 1 \ \cdot 10^{-5})$, average gap and branching nodes obtained by A_{E2TRP} are 23.79 and 6815.1, respectively vs. 52.7 and 16054.4 for A_{CC}.

Summarizing the results of the first stage, we conclude that algorithm A_{CC} performs well on the small instances of 20 nodes under mild threshold values $\mathfrak{F}$. However, when the value $\mathfrak{F}$ becomes more tight, our algorithm A_{E2TRP} shows better results.

Table 2 reflects the results of the second stage for instances with $|V| = 30$. As it follows from the table, our algorithm is able to find feasible solutions for all instances regardless of their γ and $\mathfrak{F}$ values. However, the number of instances that are solved to optimality are decreased with the γ growth and $\mathfrak{F}$ tightening.

In order examine asymptotic behavior of the proposed algorithm, we proceed with the third stage of our experiment. As it follows from Tables 3 and 4, increasing of the time limit leads to the notably better results.

TAB. 1. First stage results: comparison of algorithms A_{CC} and A_{E2TRP} on instances with $|V| = 20$ nodes: γ is the graph density, $\mathfrak{F}$ is the failure probability threshold,
OPT is the number of instances solved to optimality, ABN is the average number of branching nodes, avg gap,% is the average gap value $= (UB - LB)/LB$, time limit = 1 hour

		A_{CC}			A_{E2TRP}		
γ	$\mathfrak{F}$	OPT	ABN	avg gap,%	OPT	ABN	avg gap,%
0.25	$1 \cdot 10^{-1}$	10	1.0	0.0	10	3607.8	0.0
	$5 \cdot 10^{-2}$	10	1.0	1.6	10	3607.8	0.0
	$1 \cdot 10^{-2}$	10	1.6	0.0	10	2068.6	0.0
	$5 \cdot 10^{-3}$	10	8.5	0.0	10	1552.4	0.0
	$1 \cdot 10^{-3}$	**10**	291.5	**0.0**	8	6667.9	4.44
	$5 \cdot 10^{-4}$	**10**	227.9	**0.0**	9	6821.3	4.01
	$1 \cdot 10^{-4}$	**8**	6634.6	43.8	6	11819.2	**10.81**
	$5 \cdot 10^{-5}$	**9**	7930.3	**11.6**	6	7276.2	13.82
	$1 \cdot 10^{-5}$	**8**	9357.0	47.9	3	8025.9	**23.42**
	$5 \cdot 10^{-6}$	**5**	12823.5	67.9	2	7682.7	**29.52**
	$1 \cdot 10^{-6}$	**3**	17148.1	59.6	2	7511.1	**32.95**
0.5	$1 \cdot 10^{-1}$	**10**	1.0	**0.0**	9	9028.0	0.09
	$5 \cdot 10^{-2}$	10	1.0	0.0	10	9182.6	0.0
	$1 \cdot 10^{-2}$	**10**	1.0	**0.0**	8	4900.4	2.43
	$5 \cdot 10^{-3}$	**10**	210.3	**0.0**	8	3083.6	3.52
	$1 \cdot 10^{-3}$	**10**	159.0	**0.0**	6	4761.2	12.47
	$5 \cdot 10^{-4}$	**10**	129.8	**0.0**	6	4469.6	12.2
	$1 \cdot 10^{-4}$	**10**	205.0	**0.0**	6	4233.6	14.57
	$5 \cdot 10^{-5}$	**10**	539.2	**0.0**	6	3738.1	14.81
	$1 \cdot 10^{-5}$	**8**	16054.4	52.7	3	6815.1	**23.79**
	$5 \cdot 10^{-6}$	**8**	16364.5	52.5	2	6154.3	**30.07**
	$1 \cdot 10^{-6}$	**8**	17542.9	53.9	1	4828.3	**39.12**
0.75	$1 \cdot 10^{-1}$	**10**	1.0	**0.0**	4	23877.0	15.34
	$5 \cdot 10^{-2}$	**10**	1.0	**0.0**	4	24516.4	15.1
	$1 \cdot 10^{-2}$	**10**	1.0	**0.0**	4	10493.6	19.38
	$5 \cdot 10^{-3}$	**10**	1.6	**0.0**	5	5289.8	20.35
	$1 \cdot 10^{-3}$	**10**	6.8	**0.0**	3	5496.2	23.54
	$5 \cdot 10^{-4}$	**10**	6.2	**0.0**	3	5496.8	23.79
	$1 \cdot 10^{-4}$	**9**	1104.4	50.1	4	3566.2	**33.14**
	$5 \cdot 10^{-5}$	**8**	2728.4	46.5	2	3482.6	**43.82**
	$1 \cdot 10^{-5}$	**7**	3280.9	50.5	2	2890.6	**46.94**
	$5 \cdot 10^{-6}$	**7**	3114.2	**51.2**	1	2976.0	52.21
	$1 \cdot 10^{-6}$	**6**	2069.5	53.6	0	3565.3	**53.01**

Thus, the number of instances solved to optimality has increased in average by 6.36% for the instances of $(20, 0.25)$, by 15.45% for $(20, 0.5)$, by 12.73% for $(20, 0.75)$, by 22.72% for $(30, 0.25)$, by 5.45% for $(30, 0.5)$, and by 3.63% for $(30, 0.75)$. In particular, allowing more computation time implies the ability to solve to optimality several complex instances within the following groups (see Fig.1): $(30, 0.5, 5 \cdot 10^{-5})$, $(30, 0.5, 1 \cdot 10^{-5})$, $(30, 0.75, 1 \cdot 10^{-1})$, $(30, 0.75, 5 \cdot 10^{-2})$, $(30, 0.75, 1 \cdot 10^{-2})$, $(30, 0.75, 5 \cdot 10^{-3})$.

In turn, the average gap values have decreased by 5.59% for the instances of $(20, 0.25)$, by 8.35% for $(20, 0.5)$, by 11.98% for $(20, 0.75)$, by 10.77% for $(30, 0.25)$, by 13.82% for $(30, 0.5)$, and by 14.94% for $(30, 0.75)$. As it

TAB. 2. Second stage results: numerical evaluation of A_{E2TRP} with $|V| = 30$ nodes: γ is the graph density, $\mathfrak{F}$ is the failure threshold,
OPT is the number of instances solved to optimality, ABN is the average number of branching nodes, avg gap,% is the average gap value, %, time limit = 1 hour

	A_{E2TRP}								
	$\gamma = 0.25$			$\gamma = 0.5$			$\gamma = 0.75$		
$\mathfrak{F}$	OPT	ABN	avg gap,%	OPT	ABN	avg gap,%	OPT	ABN	avg gap,%
$1 \cdot 10^{-1}$	7	23147.2	8.28	4	19628.4	17.47	0	14430.6	58.66
$5 \cdot 10^{-2}$	7	22887.2	8.19	4	16413.2	19.47	0	13735.8	59.71
$1 \cdot 10^{-2}$	6	8513.6	8.93	3	9389.0	24.03	0	8218.4	64.04
$5 \cdot 10^{-3}$	5	4306.1	10.46	3	5160.2	27.49	0	5337.2	63.3
$1 \cdot 10^{-3}$	3	5488.1	15.38	4	4354.0	26.1	0	4594.4	68.14
$5 \cdot 10^{-4}$	3	5637.1	14.65	2	4829.4	31.8	0	4751.6	66.74
$1 \cdot 10^{-4}$	3	6545.3	29.69	1	3843.8	46.84	0	4397.2	67.22
$5 \cdot 10^{-5}$	2	6292.3	33.35	0	3686.0	54.62	0	4249.6	67.47
$1 \cdot 10^{-5}$	1	5095.4	39.04	0	4640.1	53.13	0	3146.0	72.61
$5 \cdot 10^{-6}$	1	4717.4	43.58	0	4752.1	57.35	0	1601.6	83.04
$1 \cdot 10^{-6}$	1	3964.4	46.4	0	2655.1	63.19	0	1281.0	86.1

follows from the stage three results, there is significant enhancement even for hard instances such as those ones of $|V| = 30$, $\gamma = 0.75$, and $\mathfrak{F} \leq 5 \cdot 10^{-5}$. In addition, for groups $(20, 0.25, 5 \cdot 10^{-3})$, $(20, 0.25, 1 \cdot 10^{-3})$, and $(20, 0.25, 5 \cdot 10^{-4})$, the average gaps were decreased more than 10 times (see also Fig.2).

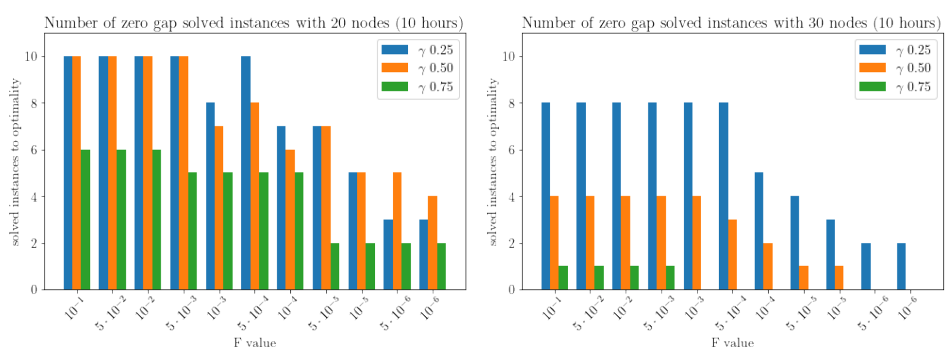

FIG. 1. The number of instances solved to optimality for (a): 20 nodes and (b): 30 nodes in 10 hours

7 Conclusion

In this paper, we introduced the Efficient 2-Terminal Reliability Problem that simulate resiliency of the supply chains under disruptions. We proposed an exact branch-and-price algorithm for this problem, relying on the classic decomposition, column generation techniques, original branching schemes, and primal heuristics. High performance of the proposed approach was confirmed by the results of the comparative numerical evaluation. We postpone

TAB. 3. Third stage results: numerical evaluation of A_{E2TRP} with $|V| = 20$ nodes: γ is the graph density, $\mathfrak{F}$ is the failure threshold, OPT is the number of instances solved to optimality, ABN is the average number of branching nodes, avg gap,% is the average gap value, %, time limit = 10 hour

		20 nodes					
		1 hour			10 hours		
γ	$\mathfrak{F}$	OPT	ABN	avg gap,%	OPT	ABN	avg gap,%
	$1 \cdot 10^{-1}$	10	3607.8	0.0	10	3607.8	0.00
	$5 \cdot 10^{-2}$	10	3607.8	0.0	10	3607.8	0.00
	$1 \cdot 10^{-2}$	10	2068.6	0.0	10	2068.6	0.00
	$5 \cdot 10^{-3}$	10	1552.4	0.0	10	1552.4	0.00
0.25	$1 \cdot 10^{-3}$	8	6667.9	4.44	8	12588.1	**1.26**
	$5 \cdot 10^{-4}$	9	6821.3	4.01	**10**	18059.1	**0.00**
	$1 \cdot 10^{-4}$	6	11819.2	10.81	**7**	73885.0	**4.85**
	$5 \cdot 10^{-5}$	6	7276.2	13.82	**7**	49897.9	**6.64**
	$1 \cdot 10^{-5}$	3	8025.9	23.42	**5**	56711.5	**11.68**
	$5 \cdot 10^{-6}$	2	7682.7	29.52	**3**	67474.4	**14.33**
	$1 \cdot 10^{-6}$	2	7511.1	32.95	**3**	59927.1	**18.75**
	$1 \cdot 10^{-1}$	9	9028.0	0.09	**10**	9182.6	**0.00**
	$5 \cdot 10^{-2}$	10	9182.6	0.0	10	9182.6	0.00
	$1 \cdot 10^{-2}$	8	4900.4	2.43	**10**	8444.4	**0.00**
	$5 \cdot 10^{-3}$	8	3083.6	3.52	**10**	6609.2	**0.00**
0.5	$1 \cdot 10^{-3}$	6	4761.2	12.47	**7**	21457.3	**3.98**
	$5 \cdot 10^{-4}$	6	4469.6	12.2	**8**	17351.9	**2.55**
	$1 \cdot 10^{-4}$	6	4233.6	14.57	6	26172.1	**5.68**
	$5 \cdot 10^{-5}$	6	3738.1	14.81	**7**	22185.0	**5.60**
	$1 \cdot 10^{-5}$	3	6815.1	23.79	**5**	29727.4	**12.81**
	$5 \cdot 10^{-6}$	2	6154.3	30.07	**5**	30477.4	**16.73**
	$1 \cdot 10^{-6}$	1	4828.3	39.12	**4**	28915.9	**22.84**
	$1 \cdot 10^{-1}$	4	23877.0	15.34	**6**	67308.0	**7.16**
	$5 \cdot 10^{-2}$	4	24516.4	15.1	**6**	67308.0	**7.16**
	$1 \cdot 10^{-2}$	4	10493.6	19.38	**6**	38357.6	**9.96**
	$5 \cdot 10^{-3}$	5	5289.8	20.35	5	28099.6	**9.15**
0.75	$1 \cdot 10^{-3}$	3	5496.2	23.54	**5**	28796.8	**10.10**
	$5 \cdot 10^{-4}$	3	5496.8	23.79	**5**	27734.8	**10.05**
	$1 \cdot 10^{-4}$	4	3566.2	33.14	**5**	17916.6	**21.26**
	$5 \cdot 10^{-5}$	2	3482.6	43.82	2	24889.3	**27.70**
	$1 \cdot 10^{-5}$	2	2890.6	46.94	2	22787.0	**33.44**
	$5 \cdot 10^{-6}$	1	2976.0	52.21	**2**	20084.4	**39.91**
	$1 \cdot 10^{-6}$	0	3565.3	53.01	**2**	20747.1	**38.87**

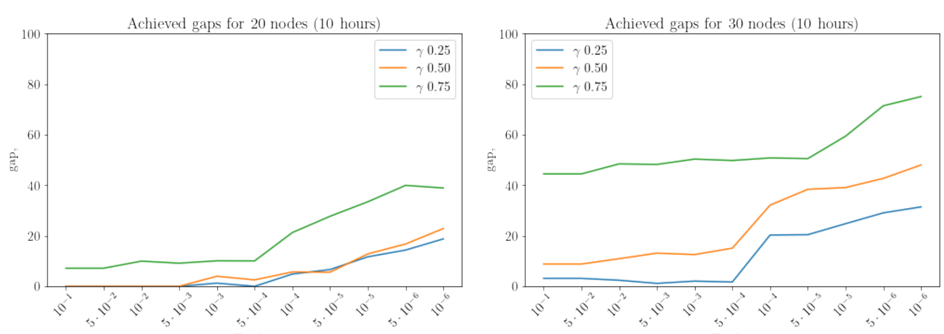

FIG. 2. Average gaps for (a): 20 and (b): 30 nodes and time limit of 10 hours

TAB. 4. Third stage results: numerical evaluation of A_{E2TRP} with $|V| = 30$ nodes: γ is the graph density, $\mathfrak{F}$ is the failure threshold, OPT is the number of instances solved to optimality, ABN is the average number of branching nodes, avg gap,% is the average gap value, %, time limit = 10 hour

		30 nodes					
		1 hour			10 hours		
γ	$\mathfrak{F}$	OPT	ABN	avg gap,%	OPT	ABN	avg gap,%
0.25	$1 \cdot 10^{-1}$	7	23147.2	8.28	**8**	56920.6	**3.14**
	$5 \cdot 10^{-2}$	7	22887.2	8.19	**8**	56920.6	**3.14**
	$1 \cdot 10^{-2}$	6	8513.6	8.93	**8**	25404.4	**2.40**
	$5 \cdot 10^{-3}$	5	4306.1	10.46	**8**	19153.5	**1.17**
	$1 \cdot 10^{-3}$	3	5488.1	15.48	**8**	26404.5	**2.06**
	$5 \cdot 10^{-4}$	3	5637.1	14.65	**8**	25947.3	**1.70**
	$1 \cdot 10^{-4}$	3	6545.3	29.69	**5**	40336.6	**20.28**
	$5 \cdot 10^{-5}$	2	6292.3	33.35	**4**	42185.3	**20.42**
	$1 \cdot 10^{-5}$	1	5095.4	39.04	**3**	36239.0	**24.77**
	$5 \cdot 10^{-6}$	1	4717.4	43.58	**2**	34515.2	**29.05**
	$1 \cdot 10^{-6}$	1	3964.4	46.4	**2**	27162.4	**31.41**
0.5	$1 \cdot 10^{-1}$	4	19628.4	17.47	**4**	52933.4	**8.82**
	$5 \cdot 10^{-2}$	4	16413.2	19.47	4	52933.4	**8.82**
	$1 \cdot 10^{-2}$	3	9389.0	24.03	**4**	36123.0	**10.92**
	$5 \cdot 10^{-3}$	3	5160.2	27.49	**4**	27190.2	**13.12**
	$1 \cdot 10^{-3}$	4	4354.0	26.1	4	23955.8	**12.56**
	$5 \cdot 10^{-4}$	2	4829.4	31.8	**3**	28477.6	**15.08**
	$1 \cdot 10^{-4}$	1	3843.8	46.84	**2**	25629.1	**32.10**
	$5 \cdot 10^{-5}$	0	3686.0	54.62	**1**	25899.9	**38.36**
	$1 \cdot 10^{-5}$	0	4640.1	53.13	**1**	32232.9	**39.06**
	$5 \cdot 10^{-6}$	0	4752.1	57.35	0	35114.3	**42.68**
	$1 \cdot 10^{-6}$	0	2655.1	63.19	0	21523.5	**47.99**
0.75	$1 \cdot 10^{-1}$	0	14430.6	58.66	**1**	57790.2	**44.45**
	$5 \cdot 10^{-2}$	0	13735.8	59.71	**1**	57810.4	**44.44**
	$1 \cdot 10^{-2}$	0	8218.4	64.04	**1**	39927.0	**48.41**
	$5 \cdot 10^{-3}$	0	5337.2	63.3	**1**	30397.0	**48.20**
	$1 \cdot 10^{-3}$	0	4594.4	68.14	0	31808.6	**50.30**
	$5 \cdot 10^{-4}$	0	4751.6	66.74	0	32276.6	**49.75**
	$1 \cdot 10^{-4}$	0	4397.2	67.22	0	31572.8	**50.76**
	$5 \cdot 10^{-5}$	0	4249.6	67.47	0	31387.6	**50.50**
	$1 \cdot 10^{-5}$	0	3146.0	72.61	0	24130.2	**59.37**
	$5 \cdot 10^{-6}$	0	1601.6	83.04	0	15438.0	**71.39**
	$1 \cdot 10^{-6}$	0	1281.0	86.1	0	12517.4	**75.03**

the study of valid inequalities, implementation of the cutting planes their heuristic separation techniques to the future work.

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