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ON $\mathcal{SSH}$-SUBGROUPS OF FINITE GROUPS

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Abstract: In this paper, we investigate the influence of $\mathcal{SSH}$-subgroups on the structure of finite groups and some new results on the p-nilpotency and p-supersolvability of finite groups are obtained.

Keywords: s-permutable, $\mathcal{SSH}$-subgroups, p-supersolvable, p-nilpotent.

1 Introduction

All groups mentioned in this paper are considered to be finite. Most of terminologies and notations are standard. The reader is referred to [9] and [10]. G always denotes a finite group and $|G|$ is the order of G. A group G is said to be p-supersolvable if all chief factors of G having order divisible by p are exactly of order p. It is known that the class of all p-supersolvable groups is a saturated formation.

Let H be a subgroup of G. H is said to be s-permutable or s-quasinormal in G, if H permutes with all Sylow subgroups of G (see [11]); H is said to be C-normal in G if G has a normal subgroup T such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the normal core of H in G (see [14]); H is said to be an $\mathcal{H}$-subgroup of G if $H^g \cap N_G(H) \leq H$ for all $g \in G$ (see [5]); H is called a weakly $\mathcal{H}$-subgroup of G if it has a normal subgroup T such that

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$G = HT$ and $H \cap T$ is an $\mathcal{H}$-subgroup of G (see [1]); H is said to be weakly $\mathcal{H}$-embedded in G if G has a normal subgroup T such that $H^G = HT$ and $H \cap T$ is an $\mathcal{H}$-subgroup of G, where H^G is the normal clousre of H in G (see [3]); H is called an $\mathcal{H}C$-subgroup of G if there exists a normal subgroup T of G such that $G = HT$ and $H^g \cap N_T(H) \leq H$ for all $g \in G$ (see [15]); H is said to be weakly $\mathcal{H}C$-embedded in G if G has a normal subgroup T such that $H^G = HT$ and $H^g \cap N_T(H) \leq H$ for all $g \in G$ (see [2]). Using these concepts, many interesting results on the structure of finite groups have been obtained in [1, 2, 3, 5, 14, 15, 16].

More recently, T. M. Al-Gafri and S. K. Nauman [7] introduced a new subgroup embedding property that extends all the above mentioned concepts as follows:

Definition 1. *A subgroup H of G is said to be an $\mathcal{SSH}$-subgroup in G if there exists an s-permutable subgroup T of G such that $H^{sG} = HT$ and $H^g \cap N_T(H) \leq H$ for all $g \in G$, where H^{sG} is the intersection of all s-permutable subgroups of G containing H.*

In [7], the authors studied the structure of finite groups under the assumption that certain subgroups of prime power orders are $\mathcal{SSH}$-subgroups. In this paper, we continue the work and present some sufficient conditions for a group to be p-supersolvable and p-nilpotent.

2 Preliminaries

Lemma 1 ([7, Lemma 2.4])**.** *Suppose that H is an $\mathcal{SSH}$-subgroup in G.*

(1) *If $H \leq K \leq G$, then H is an $\mathcal{SSH}$-subgroup in K.*

(2) *If $N \trianglelefteq G$ and $N \leq H \leq G$, then H/N is an $\mathcal{SSH}$-subgroup in G/N.*

(3) *If H is a p-subgroup and N is a normal p'-subgroup of G, then HN and HN/N are $\mathcal{SSH}$-subgroups in G and G/N, respectively.*

Lemma 2 ([5, Theorem 6 (2)])**.** *Let H be an $\mathcal{H}$-subgroup of G. If H is subnormal in G, then H is normal in G.*

Lemma 3 ([6] and [11])**.** *Suppose that H be a subgroup of G and H is s-permutable in G. Then*

(1) *H is subnormal in G.*

(2) *If $K \leq G$ and K is s-permutable in G, then $H \cap K$ is s-permutable in G.*

(3) *H/H_G is nilpotent.*

Lemma 4 ([13, Theorem A])**.** *If P is an s-permutable p-subgroup of G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 5 ([7, Theorem 3.1])**.** *Let P be a Sylow p-subgroup of a group G, for some prime p. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and every maximal subgroup of P is an $\mathcal{SSH}$-subgroup in G.*

Lemma 6. *Let H be a p-subgroup of a group G for some prime p. If N is normal in G and $(|N|, p) = 1$, then $N_{G/N}(HN/N) = N_G(H)N/N$.*

Proof. By [7, Lemma 2.3], we have $N_G(HN) = N_G(H)N$. Consequently, $N_{G/N}(HN/N) = N_G(HN)/N = N_G(H)N/N$. □

Lemma 7 ([4, Theorem 2.1.6])**.** *If G is p-supersoluble and $O_{p'}(G) = 1$, then the Sylow p-subgroup of G is normal in G.*

Lemma 8 ([12, Theorem 1.3])**.** *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$. If every maximal subgroup of P has a p-nilpotent supplement in G, then G is p-nilpotent.*

Lemma 9 ([8, Theorem 8.3.1])**.** *Let P be a Sylow p-subgroup of G, where p is an odd prime divisor of $|G|$. Then G is p-nilpotent if and only if $N_G(Z(J(P)))$ is p-nilpotent, where $J(P)$ is the Thompson subgroup of P.*

Lemma 10. *Suppose that $N \trianglelefteq G$ and $H \leq G$. If $N \leq \Phi(H)$, then $N \leq \Phi(G)$.*

Proof. Assume that $N \not\leq \Phi(G)$. Then G has a maximal subgroup M such that $N \not\leq M$ and $G = NM$. Hence $H = H \cap NM = N(H \cap M) = \Phi(H)(H \cap M) = H \cap M$. It follows that $H \leq M$ and so $N \leq M$, a contradiction. □

3 Main results

Theorem 1. *Let P be a Sylow p-subgroup of a p-solvable G for some prime p. If every maximal subgroup of P is an $\mathcal{SSH}$-subgroup in G, then G is p-supersolvable.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. Then:

(1) $O_{p'}(G) = 1$.

Assume that $O_{p'}(G) \neq 1$. Note that $P/O_{p'}(G)$ is a Sylow p-subgroup of $G/O_{p'}(G)$, where $G/O_{p'}(G)$ is p-solvable. Let $M/O_{p'}(G)$ be a maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$. Then $M = (M \cap P)O_{p'}(G)$, where $M \cap P$ is a maximal subgroup of P. Since $M \cap P$ is an $\mathcal{SSH}$-subgroup in G and $O_{p'}(G)$ is a normal p'-subgroup of G, then $M/O_{p'}(G) = (M \cap P)O_{p'}(G)/O_{p'}(G)$ is an $\mathcal{SSH}$-subgroup in $G/O_{p'}(G)$ by Lemma 1(3). Therefore $G/O_{p'}(G)$ is p-supersolvable by the minimal choice of G. Consequently, G is p-supersolvable, a contradiction.

(2) G has a unique minimal normal subgroup N such that N is an elementary abelian p-group and G/N is p-supersolvable.

Since G is p-solvable, we may assume that G has a minimal normal subgroup N. In view of step (1), N is an elementary abelian p-group and so $N \leq P$. If $N = P$, then G/N is a p'-group and hence G/N is p-supersolvable. Now assume that $N < P$. Clearly, P/N is a Sylow p-subgroup of G/N. Let M/N be a maximal subgroup of P/N. Then M is a maximal subgroup of P. By hypothesis, M is an $\mathcal{SSH}$-subgroup in G. Lemma 1(2) implies that M/N is an $\mathcal{SSH}$-subgroup in G/N. Therefore the maximal subgroups of P/N are $\mathcal{SSH}$-subgroups in G/N. Thus G/N is p-supersolvable by the minimal choice of G. If N_1 and N_2 are two distinct minimal normal subgroups of G such that $N_1, N_2 \leq P$, then G/N_1 and G/N_2 are p-supersolvable by the above proof. Since G is isomorphic to a subgroup of $G/N_1 \times G/N_2$, it follows that G is p-supersolvable. This contradiction shows that N is a unique minimal normal subgroup of G.

(3) $|N| > p$, $\Phi(G) = 1$ and $N = F(G) = O_p(G)$.

If $\Phi(G) > 1$, then $N \leq \Phi(G)$ and $G/\Phi(G) \cong (G/N)/(\Phi(G)/N)$ is p-supersolvable by step (2). Since the class of all p-supersolvable groups is saturated, it follows that G is p-supersolvable. This contradiction shows that $\Phi(G) = 1$. Since the Fitting subgroup of a group with unit Frattini subgroup coincides with the product of all abelian minimal normal subgroups, we have $N = F(G) = O_p(G)$. If $|N| = p$, then G/N is p-supersolvable by step (2) and so G is p-supersolvable, a contradiction.

(4) There exists a maximal subgroup R of P such that $N \nsubseteq R$.

Assume that $N \leq \Phi(P)$. Then $N \leq \Phi(G) = 1$ by Lemma 10, a contradiction. Hence $N \nleq \Phi(P)$ and there exists a maximal subgroup R of P such that $N \nsubseteq R$.

(5) $R \cap N \neq 1$.

Assume that $R \cap N = 1$. Then $|N| = |N|/|R \cap N| = |NR/R| = |P/R| = p$ by step (4), a contradiction.

(6) $R \cap N$ is not normal in G.

If not, we have $R \cap N = 1$ or $R \cap N = N$ by the minimal normality of N, which contradicts (4) and (5).

(7) There exists an s-permutable subgroup T of G such that RT is s-permutable in G and $R^g \cap N_T(R) \leq R$ for all $g \in G$, where $T > 1$.

By the hypothesis of the theorem, R is an $\mathcal{SSH}$-subgroup in G. Therefore G has an s-permutable subgroup T such that $R^{sG} = RT$ and $R^g \cap N_T(R) \leq R$ for all $g \in G$. By Lemma 3(2), RT is s-permutable in G. If $T = 1$, then R is s-permutable in G. By Lemma 4, $O^p(G) \leq N_G(R)$. It follows that R is normal in $PO^p(G) = G$. Consequently, $R \cap N$ is a normal subgroup of G, which contradicts (6).

(8) $N \nsubseteq T$.

Assume that $N \leq T$. Noticing that N is abelian, we have $(R \cap N)^g \cap N_G(R \cap N) = R^g \cap N \cap N_G(R \cap N) = R^g \cap N = R^g \cap N \cap T \cap P \leq R^g \cap N \cap T \cap N_G(R) = R^g \cap N \cap N_T(R)$. By step (7), we have $(R \cap N)^g \cap N_G(R \cap N) \leq R \cap N$. This shows that $R \cap N$ is an $\mathcal{H}$-subgroup of G. Obviously, $R \cap N$ is subnormal in G. In view of Lemma 2, $R \cap N$ is normal in G, which contradicts (6).

(9) T is a p-group.

Since T is s-permutable in G, it follows from Lemma 3(3) that T/T_G is nilpotent. If $T_G \neq 1$, then $N \leq T_G$ by step (2). Consequently, $N \leq T$, which contradicts (8). Hence $T_G = 1$ and T is nilpotent. In view of Lemma 3(1), $T \lhd \lhd G$. Let $T_{p'}$ be the normal Hall p'-subgroup of T. Obviously, $T_{p'} \lhd \lhd G$. Hence $T_{p'} \leq O_{p'}(G) = 1$. This implies that T is a p-group.

(10) The final contradiction.

By the maximality of R in P, $RT = R$ or P. Since RT is s-permutable in G, we have $RT \trianglelefteq PO^p(G) = G$ by Lemma 4. If $RT = R$, then $N \leq R$ by step (2), which contradicts (4). If $RT = P$, then $N = P = O_p(G)$ is elementary abelian by step (2) and so $T \trianglelefteq P$. Furthermore, $T \trianglelefteq PO^p(G) = G$ by Lemma 4. By step (2), $N = T$, contrary to step (8). $\square$

Theorem 2. *Let P be a Sylow p-subgroup of a group G, for some odd prime p. Then G is p-nilpotent if and only if every maximal subgroup P_1 of P not having a p-nilpotent supplement in G is an $\mathcal{SSH}$-subgroup in G and $N_G(P_1)$ is p-nilpotent.*

Proof. If G is p-nilpotent, then every maximal subgroup P_1 of P not having a p-nilpotent supplement in G is an $\mathcal{SSH}$-subgroup in G by [7, Lemma 2.5] and $N_G(P_1)$ is p-nilpotent. For the converse, we suppose that the result is false and let G be a counterexample of minimal order.

(1) If $P \leq K < G$, then K is p-nilpotent.

Let M be a maximal subgroup of P not having a p-nilpotent supplement in K. If M has a p-nilpotent supplement L in G, then M has a p-nilpotent supplement $L \cap K$ in K, a contradiction. Thus M is a maximal subgroup of P not having a p-nilpotent supplement in G. By hypothesis, M is an $\mathcal{SSH}$-subgroup in G. By Lemma 1(1), M is an $\mathcal{SSH}$-subgroup in K. Since $N_K(M) = K \cap N_G(M)$ and $N_G(M)$ is p-nilpotent, it follows that $N_K(M)$ is p-nilpotent. Therefore, K satisfies the hypothesis of the theorem, and so K is p-nilpotent by the minimal choice of G.

(2) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, we consider $G/O_{p'}(G)$. Let $M/O_{p'}(G)$ be a maximal subgroup of the Sylow p-subgroup $PO_{p'}(G)/O_{p'}(G)$ of $G/O_{p'}(G)$ not having a p-nilpotent supplement in $G/O_{p'}(G)$. Then $M = (M \cap P)O_{p'}(G)$, where $M \cap P$ is a maximal subgroup of P. If $M \cap P$ has a p-nilpotent supplement L in G, then $M/O_{p'}(G)$ has a p-nilpotent supplement $LO_{p'}(G)/O_{p'}(G)$ in $G/O_{p'}(G)$, a contradiction. Thus $M \cap P$ is a maximal subgroup of P not having a p-nilpotent supplement in G. By hypothesis, $M \cap P$ is an $\mathcal{SSH}$-subgroup in G. By Lemma 1(3) $M/O_{p'}(G) = (M \cap P)O_{p'}(G)/O_{p'}(G)$ is an $\mathcal{SSH}$-subgroup in $G/O_{p'}(G)$. In view of Lemma 6,

$$
\begin{aligned}
N_{G/O_{p'}(G)}(M/O_{p'}(G)) &= N_{G/O_{p'}(G)}((M \cap P)O_{p'}(G)/O_{p'}(G)) \\
&= N_G(M \cap P)O_{p'}(G)/O_{p'}(G)
\end{aligned}
$$

Since $N_G(M \cap P)O_{p'}(G)/O_{p'}(G) \cong N_G(M \cap P)/(N_G(M \cap P) \cap O_{p'}(G))$ and $N_G(M \cap P)$ is p-nilpotent, it follows that $N_{G/O_{p'}(G)}(M/O_{p'}(G))$ p-nilpotent. Therefore $G/O_{p'}(G)$ satisfies the hypothesis of the theorem. The minimal choice of G yields that $G/O_{p'}(G)$ is p-nilpotent, and so G is p-nilpotent, a contradiction.

(3) $O_p(G) \neq 1$.

Since G is not p-nilpotent, it follows from Lemma 9 that $N_G(Z(J(P))$ is not p-nilpotent, where $J(P)$ is the Thompson subgroup of P. Noticing that $Z(J(P))$ is a characteristic subgroup of P, $P \leq N_G(Z(J(P)))$. In view of step (1), we have $N_G(Z(J(P))) = G$ and so $O_p(G) \neq 1$.

(4) Let L be a normal p-subgroup of G such that $1 < L < P$, then G/L is p-nilpotent.

It is easy to see that the hypothesis of the theorem holds for G/L by Lemma 1(2). Hence G/L is p-nilpotent.

(5) G is p-solvable.

If $O_p(G) = P$, then G is p-solvable obviously. If $O_p(G) < P$, then $G/O_p(G)$ is p-nilpotent by step (4). Consequently, G is p-solvable.

(6) G has a unique minimal normal subgroup N and G/N is p-nilpotent.

Let N be a minimal normal subgroup of G. Then by steps (2) and (5), $N \leq O_p(G)$. If $N = P$, then G/N is a p'-group and hence G/N is p-nilpotent. Now assume that $N < P$. In view of step (4), G/N is p-nilpotent. If N_1 and N_2 are two distinct minimal normal subgroups of G such that $N_1, N_2 \leq P$, then G/N_1 and G/N_2 are p-nilpotent by the above proof. Since G is isomorphic to a subgroup of $G/N_1 \times G/N_2$, it follows that G is p-nilpotent. This contradiction shows that N is a unique minimal normal subgroup of G.

(7) $\Phi(G) = 1$ and $N = F(G) = O_p(G)$.

If $\Phi(G) > 1$, then $N \leq \Phi(G)$ and $G/\Phi(G) \cong (G/N)/(\Phi(G)/N)$ is p-nilpotent by step (6). Since the class of all p-nilpotent groups is saturated, it follows that G is p-nilpotent. This contradiction shows that $\Phi(G) = 1$. Since the Fitting subgroup of a group with unit Frattini subgroup coincides with the product of all abelian minimal normal subgroups, we have $N = F(G) = O_p(G)$.

(8) $|N| > p$.

If $|N| = p$, then G/N is p-nilpotent by step (6) and so G is p-supersolvable. In view of Lemma 7, P is normal in G and so $P = O_p(G)$. By step (7), $|P| = p$. Then the maximal subgroup of P is 1. By the hypothesis of the theorem, 1 has a p-nilpotent supplement G or $N_G(1) = G$ is p-nilpotent, a contradiction.

(9) N has a p-nilpotent supplement M in G.

Since $\Phi(G) = 1$ by step (7), it follows that G has a maximal subgroup M such that $N \nleq M$ and so $G = NM$. It is easy to see that $N \cap M$ is normal in G. Consequently, $N \cap M = 1$ and $M \cong G/N$ is p-nilpotent by step (6).

(10) There exists a maximal subgroup R of P such that R has no p-nilpotent supplement in G. Then R is an $\mathcal{SSH}$-subgroup in G by hypothesis.

This follows from Lemma 8.

(11) $N \nleq R$.

If $N \leq R$, then R has a p-nilpotent supplement M in G by step (9), contrary to step (10).

(12) The final contradiction.

See the similar arguments to those used in the proof of Theorem 1. $\square$

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