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СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Vol. 22, No. 2, pp. 928–932 (2025) УДК 512.542
https://doi.org/10.33048/semi.2025.22.058 MSC 20E45

ON GROUPS WITH THE SAME SET OF CONJUGACY CLASS SIZES AS NILPOTENT GROUPS

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Communicated by I.B. Gorshkov

Abstract: We construct examples of groups which have the same set of conjugacy class sizes as nilpotent groups, while having a trivial centre. This answers a question posed by A. R. Camina in 2006.

Keywords: finite group, nilpotent group, conjugacy class.

1 Introduction

Let G be a finite group. In [1], Baer defined the index of x in G, denoted by $\mathrm{Ind}_G(x)$, as $|G : \mathrm{C}_G(x)|$, which represents the size of the conjugacy class of G containing x. In [11], Itô defined the conjugate type vector of G as $(n_1, n_2, \dots, n_r)$, where $n_1 > n_2 > \dots > n_r = 1$ are the indices of all elements in G. Since we are not interested in the ordering of these indices, we will denote the set of indices (sizes of conjugacy classes) by $\mathrm{N}(G)$, i.e., $\mathrm{N}(G) = \{n_1, n_2, \dots, n_r\}$.

Many authors have studied the relationship between the structure of finite groups and the sizes of their conjugacy classes. Itô proved that if $\mathrm{N}(G) = \{1, n\}$, then G must be the direct product of a p-group and an abelian p'-group [11]. Ishikawa proved that the nilpotent class of such groups is at most 3 [10]. More results can be found in [4].

Zhou Wei, On Groups with the Same Set of Conjugacy Class Sizes as Nilpotent Groups.

Received February, 26, 2025, Published September, 15, 2025.

It is easy to see that if G is nilpotent, then $\mathrm{N}(G) = \mathrm{N}(P_1) \times \mathrm{N}(P_2) \times \ldots \times \mathrm{N}(P_k)$, where $P_1, P_2, \ldots, P_k$ are the Sylow subgroups of G. A natural question is whether the converse holds:

Question 1 ([3, Question 1]). *Let G and H be finite groups with H being nilpotent. Suppose G and H have the same sets of conjugacy class sizes, is G nilpotent?*

In [6], Cossey proved that every finite set of p-powers containing 1 can be a set of conjugacy class sizes of some p-group. Therefore, the above question can be restated as follows: If $\mathrm{N}(G) = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_r$, where Ω_i is a finite set of p_i-powers containing 1 and $p_1, p_2, \ldots, p_r$ are distinct primes, is G nilpotent? The answer is positive in some special cases. For example, if $\mathrm{N}(G) = \{1, p_1^{m_1}\} \times \{1, p_2^{m_2}\} \times \cdots \times \{1, p_k^{m_k}\}$, where $p_1^{m_1}, p_2^{m_2}, \ldots, p_k^{m_k}$ are powers of distinct primes, then G is nilpotent [5]. More generally, if $\mathrm{N}(G) = \{1, n_1\} \times \{1, n_2\} \times \cdots \times \{1, n_r\}$, where $n_1, n_2, \ldots, n_r$ are pairwise coprime integers, then G is nilpotent [9]. A more general question is as follows:

Question 2 ([8, Question 0.1]). *Let G be a group such that $\mathrm{N}(G) = \Omega \times \Delta$. Which Δ and Ω guarantee that $G \cong A \times B$, where A and B are subgroups such that $\mathrm{N}(A) = \Omega$ and $\mathrm{N}(B) = \Delta$?*

However, the answer to Question 1 is not always positive, as some counterexamples are provided in [3]. In that paper, A. R. Camina posed a number of questions about the structure of groups with the same set of conjugacy class sizes as nilpotent groups. One of them goes as follows:

Question 3 ([3, Question 4]). *Let G and H be finite groups with H nilpotent. Suppose $\mathrm{N}(G) = \mathrm{N}(H)$, but G is not nilpotent. Does G have a nontrivial centre?*

Using GAP[7], we find that Question 3 does not have a positive answer in general. The smallest counterexamples are two groups of order $486 = 3^5 \times 2$, with the set of conjugacy class sizes $\{1, 3, 27\} \times \{1, 2\}$. One of them is SmallGroup(486, 36), and the other is SmallGroup(486, 38). Moreover, we constructed the following series of counterexamples.

Main Theorem. *Let p and q be primes such that $p = 2q + 1$. Let $G = H \rtimes (A \rtimes B)$, where H, A and B are defined as follows:*

1) $H = K/N$, where $K = \langle k_1 \rangle \times \langle k_2 \rangle \times \ldots \times \langle k_p \rangle$ is the direct product of p cyclic groups of order p, and $N = \langle k_1 k_2 \ldots k_p \rangle$;

2) $A \rtimes B$ is a subgroup of the symmetric group Sym_p: $A = \langle \alpha \rangle$ and $B = \langle \beta \rangle$, where $\alpha = (12 \ldots p)$ and $\beta = (m_1 \ldots m_q)(n_1 \ldots n_q)$, with $\{m_1, \ldots, m_q, n_1, \ldots, n_q\} = \{2, 3, \ldots, p\}$. Additionally, $\alpha^\beta = \alpha^r$ where $1 < r < q$ and $r^q \equiv 1 \pmod p$. For any $\gamma \in A \rtimes B$ and $k_1^{x_1} k_2^{x_2} \ldots k_p^{x_p} N \in H$, $(k_1^{x_1} k_2^{x_2} \ldots k_p^{x_p} N)^\gamma = k_{1^\gamma}^{x_1} k_{2^\gamma}^{x_2} \ldots k_{p^\gamma}^{x_p} N$.

Then $\mathrm{N}(G) = \{1, p, p^{p-2}\} \times \{1, q\}$, and $\mathrm{Z}(G) = 1$.

From this theorem, the following corollary can be derived.

Corollary. *Let p and q be primes such that $p = 2q + 1$. Let G, H and A be as defined above. Let $L = P \times Q$, where $P = H \rtimes A$ and $Q = C_{q^2} \rtimes C_q$. Then, we have* $\mathrm{N}(G) = \mathrm{N}(L)$.

A prime number q such that $2q + 1$ is also a prime is called a Sophie Germain prime. As of now, the largest known Sophie Germain prime is $2618163402417 \times 2^{1290000} - 1$ [2]. It is conjectured that there are infinitely many Sophie Germain primes, but this has not been proven. So we cannot conclude that there are infinitely many counterexamples to Question 3.

2 Preliminaries

Lemma 1. *Let G be a finite group, $H \leq G$ and $x \in G$. If n is an integer and $(n, |x|) = 1$, then $\mathrm{C}_H(x) = \mathrm{C}_H(x^n)$.*

Proof. It is clear that $\mathrm{C}_H(x) \leq \mathrm{C}_H(x^n)$. By Euler's theorem, we have $n^t \equiv 1 \pmod{|x|}$, where $t = \varphi(|x|)$. Hence $x = (x^n)^{n^{t-1}}$ and so $\mathrm{C}_H(x^n) \leq \mathrm{C}_H(x)$. Therefore $\mathrm{C}_H(x) = \mathrm{C}_H(x^n)$. □

Lemma 2. *Let $G = H \rtimes \langle a \rangle$, where H is an abelian group and $(|H|, |a|) = 1$. Then for any element h of H, $\mathrm{Ind}_G(ha) = \mathrm{Ind}_G(a) = |H : \mathrm{C}_H(a)|$.*

Proof. Let $|a| = n$. It is easy to verify that $\mathrm{C}_G(a) = \mathrm{C}_H(a)\langle a \rangle$ and $\mathrm{Ind}_G(a) = |H : \mathrm{C}_H(a)|$. Since $(ha)^n = hh^{a^{-1}} \dots h^{a^{1-n}} a^n = hh^{a^{-1}} \dots h^{a^{1-n}} \in H$, n is a divisor of $|ha|$. Let $t = |ha|/n$. We have $(ha)^t$ is an element of order n and $\langle (ha)^t \rangle$ is a complement to H in G. Hence $G = H \rtimes \langle (ha)^t \rangle$ and so $\mathrm{C}_G((ha)^t) = \mathrm{C}_H((ha)^t)\langle (ha)^t \rangle$. Since $(|H|, n) = 1$, we have $(t, n) = 1$. By Lemma 1, $\mathrm{C}_H((ha)^t) = \mathrm{C}_H(a^t) = \mathrm{C}_H(a)$. Since $\mathrm{C}_H(a)\langle (ha)^t \rangle \leq \mathrm{C}_G(ha) \leq \mathrm{C}_G((ha)^t)$, we have $\mathrm{C}_G(ha) = \mathrm{C}_H(a)\langle (ha)^t \rangle$. Therefore $\mathrm{Ind}_G(ha) = \mathrm{Ind}_G(a) = |H : \mathrm{C}_H(a)|$. □

3 Proof the main theorem

Let G, A, B, H, N, p, q be as defined in the main theorem. For convenience, we use $(x_1, x_2, \dots, x_p)$ to represent the element $k_1^{x_1} k_2^{x_2} \dots k_p^{x_p} N$ of H, $x_1, \dots, x_p \in \mathbb{N}$. Under this notation, we have $(x, x, \dots, x) = 1$, $\forall x \in \mathbb{N}$. We can always set $x_1 = 0$, in which case $x_2, \dots, x_p$ are determined. Let $h = (0, x_2, \dots, x_p) \in H$, $a \in A$, $b \in B$ and $h, a, b \neq 1$. It is clear that $|G| = p^p q$. We break the proof into the following steps.

(1) $|\mathrm{C}_H(a)| = p$ and $\mathrm{Ind}_G(a) = p^{p-2}q$.

By Lemma 1, it suffices to consider the case $a = \alpha$, i.e., when $(0, x_2, \dots, x_{p-1}, x_p)^a = (x_p, 0, x_2, \dots, x_{p-1})$. If $h \in \mathrm{C}_H(a)$, we have $0 - x_p \equiv x_2 - 0 \equiv \dots \equiv x_p - x_{p-1} \pmod{p}$. If $x_p = 1$, then $h = (0, p-1, p-2, \dots, 1)$. In fact, $\mathrm{C}_H(a) = \langle (0, p-1, p-2, \dots, 1) \rangle$. Therefore $|\mathrm{C}_H(a)| = p$.

Let $h_1 a_1 b_1 \in \mathrm{C}_G(a)$, where $h_1 \in H$, $a_1 \in A$ and $b_1 \in B$. We have $h_1 a_1 b_1 = (h_1 a_1 b_1)^a = h_1^a a_1 b_1^a = h_1^a (a_1 a^{-1} a^{b_1^{-1}}) b_1$. It follows that $h_1 \in C_H(a)$ and $b_1 = 1$. Hence $\mathrm{C}_G(a) = \mathrm{C}_H(a)A$. Thus, $|\mathrm{C}_G(a)| = p^2$ and so $\mathrm{Ind}_G(a) = p^{p-2}q$.

(2) $|\mathrm{C}_H(b)| = p^2$ and $\mathrm{Ind}_G(b) = p^{p-2}$.

It is easy to verify that $\mathrm{C}_H(b) = \langle k_{m_1} \dots k_{m_q} N, k_{n_1} \dots k_{n_q} N \rangle$. Therefore, $\mathrm{C}_H(b) = p^2$. Moreover, $\mathrm{C}_H(a) \cap \mathrm{C}_H(b) = 1$.

If $h_1 a_1 b_1 \in \mathrm{C}_G(b)$, then $h_1 a_1 b_1 = (h_1 a_1 b_1)^b = h_1^b a_1^b b_1$. It follows that $h_1 \in \mathrm{C}_H(b)$ and $a_1 = 1$. Therefore $\mathrm{C}_G(b) = \mathrm{C}_H(b) B$. We have $|\mathrm{C}_G(b)| = p^2 q$ and so $\mathrm{Ind}_G(b) = p^{p-2}$.

(3) $\mathrm{Ind}_G(ab) = p^{p-2}$.

By Sylow's theorems, AB has p Sylow q-subgroups. Since that $p(q-1) + p = pq = |AB|$, every element in $AB - A$ has order q. Hence ab must be contained in some conjugate of B. Thus, $\mathrm{Ind}_G(ab) = \mathrm{Ind}_G(b) = p^{p-2}$.

(4) $\{\mathrm{Ind}_G(h) \mid h \in H\} = \{p, q, pq\}$.

It is clear that $H \leq \mathrm{C}_G(h)$. If $h \in \mathrm{C}_G(a)$, then $\mathrm{C}_G(h) = HA$ and $\mathrm{Ind}_G(h) = q$. If $h \in \mathrm{C}_G(b)$ or $\mathrm{C}_G(ab)$, then $\mathrm{C}_G(h) = HB$ or $H\langle ab \rangle$ and $\mathrm{Ind}_G(h) = p$. The number of such h in all the cases above is at most $|\mathrm{C}_H(a)| + p|\mathrm{C}_H(b)| = p^3 + p$. By our initial assumption, p must be greater than or equal to 5, so $p^3 + p < p^{p-1} = |H|$. Hence there exists $h \in H$ such that $\mathrm{C}_G(H) = H$. For such h, $\mathrm{Ind}_G(h) = pq$.

(5) $\mathrm{Ind}_G(ha) = p^{p-2} q$.

Let $h_1 a_1 b_1 \in \mathrm{C}_G(ha)$, where $h_1 \in H$, $a_1 \in A$ and $b_1 \in B$. We have $ha = (ha)^{h_1 a_1 b_1} = (h h_1^{-1} h_1^{a^{-1}})^{a_1 b_1} a^{b_1}$. Hence $b_1 = 1$ and so $\mathrm{C}_G(ha) = \mathrm{C}_{HA}(ha)$. We have $(ha)^p = (h h^{a^{-1}} h^{a^{-2}} \dots h^{a^{1-p}}) a^p = h h^{a^{-1}} h^{a^{-2}} \dots h^{a^{1-p}}$. If $h = (x_1, x_2, \dots, x_p)$, then $h h^{a^{-1}} h^{a^{-2}} \dots h^{a^{1-p}} = (x_1 + \dots + x_p, \dots, x_1 + \dots + x_p) = 1$. Hence ha is an element of order p. Since $\langle ha \rangle \cap H = 1$, We have $HA = H \rtimes \langle ha \rangle$. Therefore $\mathrm{C}_G(ha) = \mathrm{C}_H(ha)\langle ha \rangle = \mathrm{C}_H(a)\langle ha \rangle$. Thus $|\mathrm{C}_G(ha)| = p^2$ and so $\mathrm{Ind}_G(ha) = p^{p-2} q$.

(6) $\mathrm{Ind}_G(hab) = \mathrm{Ind}_G(hb) = p^{p-2}$.

By (3), ab and b are conjugate, so hab must be conjugate to $h'b$ where h' is some element in H. Thus, we only need to consider $\mathrm{Ind}_G(hb)$. Let $h_1 a_1 b_1 \in \mathrm{C}_G(hb)$ where $h_1 \in H, a_1 \in A, b_1 \in B$. We have $hb = (hb)^{h_1 a_1 b_1} = (h h_1^{-1} h_1^{b^{-1}} b)^{a_1 b_1} = (h h_1^{-1} h_1^{b^{-1}})^{a_1 b_1} (a_1^{-1} a_1^{b^{-1}})^{b_1} b$. Hence $a_1 = 1$ and $\mathrm{C}_G(hb) = \mathrm{C}_{HB}(hb)$. It follows that $\mathrm{Ind}_G(hb) = \mathrm{Ind}_{HB}(hb) \times p$. By Lemma 2, $\mathrm{Ind}_{HB}(hb) = |H : \mathrm{C}_H(b)| = p^{p-3}$. Therefore, $\mathrm{Ind}_G(hb) = p^{p-2}$.

Throughout (1)–(6), all nontrivial elements of G have been considered, so $\mathrm{N}(G) = \{1, p, p^{p-2}\} \times \{1, q\}$ and $\mathrm{Z}(G) = 1$. The theorem is proved.

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