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THE ALGEBRAIC AND GEOMETRIC CLASSIFICATION OF JORDAN SUPERALGEBRAS

K. ABDURASULOV, **R. LUBKOV**, AND **A. SAYDALIYEV**

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Abstract: We give the algebraic and geometric classification of complex four-dimensional Jordan superalgebras. In particular, we describe all irreducible components in the corresponding varieties.

Keywords: Jordan superalgebra, orbit closure, degeneration, rigid superalgebra.

Introduction

The algebraic classification (up to isomorphism) of algebras of small dimensions from a certain variety defined by a family of polynomial identities is a classic problem in the theory of non-associative algebras. Another interesting approach to studying algebras of a fixed dimension is to take a geometric perspective—that is, to study their degenerations and deformations. The results in which the complete information about degenerations of a certain variety is obtained are generally referred to as the geometric classification of the algebras of these varieties. There are many results related to the algebraic and geometric classification of Jordan, Lie, Leibniz, Zinbiel,

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and other algebras (see, [1, 3–9, 11–13, 15–20, 24–31, 40]). Degenerations are related to deformations and cohomology [10, 14, 39].

Superalgebras emerged in physics to provide a unified framework for the study of super symmetry of elementary particles. Jordan algebras, originating from quantum mechanics, gained importance due to their close connection to Lie theory. Finite-dimensional simple Jordan superalgebras over an algebraically closed field of characteristic zero were classified by Kac [32] in 1977. One case remained unresolved, which was considered by Kantor [33] in 1990. Recently Racine and Zelmanov [38] presented a classification of finite-dimensional simple Jordan superalgebras over arbitrary fields of characteristic distinct from 2, focusing on cases where the even part is semisimple. For the opposite case, where the even part is not semisimple, a classification was obtained by Martínez and Zelmanov [37] in 2002, completing the entire project.

In [36], the authors focus on the classification of Jordan superalgebras of dimension up to three over an algebraically closed field of characteristic distinct from 2. The main goal of that paper was to determine the minimal dimension of exceptional Jordan superalgebras, a problem raised in [34]. In [21], authors obtained all four-dimensional indecomposable Jordan superalgebras.

In the present paper, we obtain algebraic and geometric classification of four-dimensional Jordan superalgebras and find all irreducible components within that variety. Section 1 outlines the foundational concepts and establishes several key preliminary results. In Section 2, we proceed to classify all four-dimensional Jordan superalgebras. Based on this classification, we determine the irreducible components within the variety in Section 3.

Our main results are summarized below.

Theorem A. *The variety of complex four-dimensional Jordan superalgebras of type* $(1,3)$ *has dimension* 7*. It is defined by* 11 *rigid superalgebras and can be described as the closure of the union of* $\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$*-orbits of the superalgebras given in Theorem 12.*

Theorem B. *The variety of complex four-dimensional Jordan superalgebras of type* $(2,2)$ *has dimension* 6*. It is defined by* 24 *rigid superalgebras and* 1 *one-parametric families of superalgebras and can be described as the closure of the union of* $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$*-orbits of the superalgebras given in Theorem 13.*

Theorem C. *The variety of complex four-dimensional Jordan superalgebras of type* $(3,1)$ *has dimension* 9*. It is defined by* 21 *rigid superalgebras and can be described as the closure of the union of* $\mathrm{GL}_3(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$*-orbits of the superalgebras given in Theorem 14.*

1 Preliminaries

1.1. Jordan superalgebras.

Definition 1. *A commutative algebra is called a Jordan algebra if it satisfies the identity*

$$(x^2y)x = x^2(yx).$$

Definition 2. *A superalgebra $\mathcal{A}$ is an algebra with a $\mathbb{Z}_2$-grading, i.e. $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ is a direct sum of two vector spaces and*

$$\mathcal{A}_i\mathcal{A}_j \subseteq \mathcal{A}_{i+j}, \quad \textit{where} \quad i, j \in \mathbb{Z}_2.$$

Let G be the Grassmann algebra over $\mathbb{F}$ given by the generators $1, \xi_1, \ldots, \xi_n, \ldots$ and the defining relations $\xi_i^2 = 0$ and $\xi_i\xi_j = -\xi_j\xi_i$. The elements $1, \xi_{i_1}\xi_{i_2}\ldots\xi_{i_k}$, $i_1 < i_2 < \ldots < i_k$, form a basis of the algebra G over $\mathbb{F}$. Denote by G_0 and G_1 the subspaces spanned by the products of even and odd lengths, respectively; then G can be represented as the direct sum of these subspaces, $G = G_0 \oplus G_1$. Here the relations $G_iG_j \subseteq G_{i+j(mod\ 2)}$, $i, j = 0, 1$, hold. In other words, G is a $\mathbb{Z}_2$-graded algebra (or a superalgebra) over $\mathbb{F}$. Suppose now that $A = A_0 \oplus A_1$ is an arbitrary superalgebra over $\mathbb{F}$. Consider the tensor product $G \otimes A$ of $\mathbb{F}$-algebras. The subalgebra

$$G(A) = G_0 \otimes A_0 + G_1 \otimes A_1$$

of $G \otimes A$ is referred to as the Grassmann envelope of the superalgebra A. Let Ω be a variety of algebras over $\mathbb{F}$. A superalgebra $A = A_0 \oplus A_1$ is referred to as an Ω-superalgebra if its Grassmann envelope $G(A)$ is an algebra in Ω. In particular, $A = A_0 \oplus A_1$ is referred to as a Jordan superalgebra if its Grassmann envelope $G(A)$ is a Jordan algebra.

Definition 3. *A Jordan superalgebra is a superalgebra $\mathfrak{J} = \mathfrak{J}_0 + \mathfrak{J}_1$ satisfying the graded identities:*

$$xy = (-1)^{|x||y|}yx,$$

$$((xy)z)t + (-1)^{|y||z|+|y||t|+|z||t|}((xt)z)y + (-1)^{|x||y|+|x||z|+|x||t|+|z||t|}((yt)z)x =$$
$$(xy)(zt) + (-1)^{|t|(|y|+|z|)}(xt)(yz) + (-1)^{|y||z|}(xz)(yt),$$

where $|x| = i$ for $x \in \mathfrak{J}_i$.

For convenience, we use the following notation in the next sections.

$$J(x, y, z, t) =$$
$$((xy)z)t + (-1)^{|y||z|+|y||t|+|z||t|}((xt)z)y + (-1)^{|x||y|+|x||z|+|x||t|+|z||t|}((yt)z)x -$$
$$(xy)(zt) - (-1)^{|t|(|y|+|z|)}(xt)(yz) - (-1)^{|y||z|}(xz)(yt).$$

Definition 4. *For arbitrary elements $x, y \in \mathfrak{J}_0 \cup \mathfrak{J}_1$ of a Jordan superalgebra, the operator $D\colon \mathfrak{J} \to \mathfrak{J}$ satisfying*

$$D(xy) = D(x)y + (-1)^{|D||x|}xD(y)$$

is a derivation of $\mathfrak{J}$, where $|D| = 0$ if D preserves the gradation and $|D| = 1$ otherwise.

In 2013, M.E.Martin [35] described all Jordan algebras up to dimension four over an algebraically closed field, and in [36] they obtained all Jordan superalgebras of dimension up to three. Based on those papers, we present the list of indecomposable Jordan algebras and superalgebras of dimension less than or equal to 3 in the following tables, as they will be used in the sequel.

Table 1: Indecomposable Jordan algebras

$\mathfrak{J}$		Multiplication table						dim
$\mathcal{U}_1$	:	$e^2=e$						1
$\mathcal{U}_2$	:	$e^2=0$						1
$\mathcal{B}_1$	:	$e_1^2=e_1$	$e_1e_2=e_2$	$e_2^2=0$				2
$\mathcal{B}_2$	:	$e_1^2=e_1$	$e_1e_2=\frac{1}{2}e_2$	$e_2^2=0$				2
$\mathcal{B}_3$	:	$e_1^2=e_2$	$e_1e_2=0$	$e_2^2=0$				2
$\mathcal{T}_1$	:	$e_1^2=e_1$	$e_2^2=e_3$	$e_3^2=0$	$e_1e_2=e_2$	$e_1e_3=e_3$	$e_2e_3=0$	3
$\mathcal{T}_2$	:	$e_1^2=e_1$	$e_2^2=0$	$e_3^2=0$	$e_1e_2=e_2$	$e_1e_3=e_3$	$e_2e_3=0$	3
$\mathcal{T}_3$	:	$e_1^2=e_2$	$e_2^2=0$	$e_3^2=0$	$e_1e_2=e_3$	$e_1e_3=0$	$e_2e_3=0$	3
$\mathcal{T}_4$	:	$e_1^2=e_2$	$e_2^2=0$	$e_3^2=0$	$e_1e_2=0$	$e_1e_3=e_2$	$e_2e_3=0$	3
$\mathcal{T}_5$	:	$e_1^2=e_1$	$e_2^2=e_2$	$e_3^2=e_1+e_2$	$e_1e_2=0$	$e_1e_3=\frac{1}{2}e_3$	$e_2e_3=\frac{1}{2}e_3$	3
$\mathcal{T}_6$	:	$e_1^2=e_1$	$e_2^2=0$	$e_3^2=0$	$e_1e_2=\frac{1}{2}e_2$	$e_1e_3=e_3$	$e_2e_3=0$	3
$\mathcal{T}_7$	:	$e_1^2=e_1$	$e_2^2=0$	$e_3^2=0$	$e_1e_2=\frac{1}{2}e_2$	$e_1e_3=\frac{1}{2}e_3$	$e_2e_3=0$	3
$\mathcal{T}_8$	:	$e_1^2=e_1$	$e_2^2=e_3$	$e_3^2=0$	$e_1e_2=\frac{1}{2}e_2$	$e_1e_3=0$	$e_2e_3=0$	3
$\mathcal{T}_9$	:	$e_1^2=e_1$	$e_2^2=e_3$	$e_3^2=0$	$e_1e_2=\frac{1}{2}e_2$	$e_1e_3=e_3$	$e_2e_3=0$	3
$\mathcal{T}_{10}$	:	$e_1^2=e_1$	$e_2^2=e_2$	$e_3^2=0$	$e_1e_2=0$	$e_1e_3=\frac{1}{2}e_3$	$e_2e_3=\frac{1}{2}e_3$	3

Below we list indecomposable superalgebras denoted by $\mathcal{S}_j^i$ where the exponent i represents its dimension.

Table 2: Indecomposable Jordan superalgebras

$\mathfrak{J}$		Multiplication table			
$\mathcal{S}_1^1$	:	$f^2=0$			
$\mathcal{S}_1^2$	:	$e^2=e$	$ef=\frac{1}{2}f$		
$\mathcal{S}_2^2$	:	$e^2=e$	$ef=f$		
$\mathcal{S}_1^3$	:	$ef_1=f_2$	$f_1f_2=e$		
$\mathcal{S}_2^3$	:	$f_1f_2=e$			
$\mathcal{S}_3^3$	:	$ef_1=f_2$			
$\mathcal{S}_4^3$	:	$e^2=e$	$ef_1=f_1$	$ef_2=\frac{1}{2}f_2$	
$\mathcal{S}_5^3$	:	$e^2=e$	$ef_1=\frac{1}{2}f_1$	$ef_2=\frac{1}{2}f_2$	
$\mathcal{S}_6^3$	:	$e^2=e$	$ef_1=f_1$	$ef_2=f_2$	
$\mathcal{S}_7^3$	:	$e^2=e$	$ef_1=\frac{1}{2}f_1$	$ef_2=\frac{1}{2}f_2$	$f_1f_2=e$
$\mathcal{S}_8^3$	:	$e^2=e$	$ef_1=f_1$	$ef_2=f_2$	$f_1f_2=e$
$\mathcal{S}_9^3$	:	$e_1^2=e_1$	$e_1e_2=e_2$	$e_1f=\frac{1}{2}f$	
$\mathcal{S}_{10}^3$	:	$e_1^2=e_1$	$e_1e_2=e_2$	$e_1f=\frac{1}{2}f$	
$\mathcal{S}_{11}^3$	:	$e_1^2=e_1$	$e_1e_2=\frac{1}{2}e_2$	$e_1f=\frac{1}{2}f$	
$\mathcal{S}_{12}^3$	:	$e_1^2=e_1$	$e_1e_2=\frac{1}{2}e_2$	$e_1f=f$	
$\mathcal{S}_{13}^3$	:	$e_1^2=e_1$	$e_2^2=e_2$	$e_1f=\frac{1}{2}f$	$e_2f=\frac{1}{2}f$

1.2. Degenerations. Given an (m,n)-dimensional vector superspace $V = V_0 \oplus V_1$, the set

$$\mathrm{Hom}(V \otimes V, V) = (\mathrm{Hom}(V \otimes V, V))_0 \oplus (\mathrm{Hom}(V \otimes V, V))_1$$

is a vector superspace of dimension $m^3 + 3mn^2$. This space has a structure of the affine variety $\mathbb{C}^{m^3+3mn^2}$. If we fix a basis $\{e_1, \ldots, e_m, f_1, \ldots, f_n\}$ of V, then any $\mu \in \mathrm{Hom}(V \otimes V, V)$ is determined by $m^3 + 3mn^2$ structure constants $\alpha_{i,j}^k, \beta_{i,j}^q, \gamma_{i,j}^q, \delta_{p,q}^k \in \mathbb{C}$ such that

$$\mu(e_i \otimes e_j) = \sum_{k=1}^{m} \alpha_{i,j}^k e_k, \quad \mu(e_i \otimes f_p) = \sum_{q=1}^{n} \beta_{i,p}^q f_q,$$

$$\mu(f_p \otimes e_i) = \sum_{q=1}^{n} \gamma_{p,i}^q f_q, \quad \mu(f_p \otimes f_q) = \sum_{k=1}^{m} \delta_{p,q}^k e_k.$$

A subset $\mathbb{L}(T)$ of $\mathrm{Hom}(V \otimes V, V)$ is *Zariski-closed* if it can be defined by a set of polynomial equations T in the variables $\alpha_{i,j}^k, \beta_{i,p}^q, \gamma_{p,i}^q, \delta_{p,q}^k$ $(1 \leq i,j,k \leq m,\ 1 \leq p,q \leq n)$.

Let $\mathcal{S}^{m,n}$ be the set of all superalgebras of dimension (m,n) defined by the family of polinomial super-identities T, understood as a subset $\mathbb{L}(T)$ of an affine variety $\mathrm{Hom}(V \otimes V, V)$. Then one can see that $\mathcal{S}^{m,n}$ is a Zariski-closed subset of the variety $\mathrm{Hom}(V \otimes V, V)$. The group $G = (\mathrm{Aut}\, V)_0 \simeq \mathrm{GL}(V_0) \oplus \mathrm{GL}(V_1)$ acts on $\mathcal{S}^{m,n}$ by conjugations:

$$(g * \mu)(x \otimes y) = g\mu(g^{-1}x \otimes g^{-1}y)$$

for $x, y \in V$, $\mu \in \mathbb{L}(T)$ and $g \in G$.

Denote by $O(\mu)$ the orbit of $\mu \in \mathbb{L}(T)$ under the action of G and by $\overline{O(\mu)}$ the Zariski closure of $O(\mu)$. Let $J, J' \in \mathcal{S}^{m,n}$ and $\lambda, \mu \in \mathbb{L}(T)$ represent J and J', respectively. We say that λ degenerates to μ and write $\lambda \to \mu$ if $\mu \in \overline{O(\lambda)}$. Note that in this case we have $\overline{O(\mu)} \subset \overline{O(\lambda)}$. Hence, the definition of a degeneration does not depend on the choice of μ and λ, and we write indistinctly $J \to J'$ instead of $\lambda \to \mu$ and $O(J)$ instead of $O(\lambda)$. If $J \not\cong J'$, then the assertion $J \to J'$ is called a *proper degeneration*. We write $J \not\to J'$ if $J' \notin \overline{O(J)}$.

Let J be represented by $\lambda \in \mathbb{L}(T)$. Then J is *rigid* in $\mathbb{L}(T)$ if $O(\lambda)$ is an open subset of $\mathbb{L}(T)$. Recall that a subset of a variety is called irreducible if it cannot be represented as a union of two non-trivial closed subsets. A maximal irreducible closed subset of a variety is called *irreducible component*. In particular, J is rigid in $\mathcal{S}^{m,n}$ iff $\overline{O(\lambda)}$ is an irreducible component of $\mathbb{L}(T)$. It is well known that any affine variety can be represented as a finite union of its irreducible components in a unique way. We denote by $\mathrm{Rig}(\mathcal{S}^{m,n})$ the set of rigid superalgebras in $\mathcal{S}^{m,n}$.

1.3. Principal notation. Let $\mathcal{JS}^{m,n}$ be the set of all Jordan superalgebras of dimension (m,n). Let J be a Jordan superalgebra with a fixed basis $\{e_1,\dots,e_m,f_1,\dots f_n\}$, defined by

$$e_ie_j=\sum_{k=1}^{m}\alpha_{ij}^ke_k,\quad e_if_j=\sum_{k=1}^{n}\beta_{ij}^kf_k,\quad f_if_j=\sum_{k=1}^{m}\gamma_{ij}^ke_k.$$

In the sequel, we use the following notation:

(1) $\mathfrak{a}(J)$ is the Jordan superalgebra with the same underlying vector superspace as J and defined by $f_if_j=\sum_{k=1}^{n}\gamma_{ij}^ke_k$.

(2) $J^1=J$, $J^r=J^{r-1}J+J^{r-2}J^2+\cdots+JJ^{r-1}$, and in every case $J^r=(J^r)_0\oplus(J^r)_1$.

(3) $c_{i,j}=\dfrac{\operatorname{tr}(L(x)^i)\cdot\operatorname{tr}(L(y)^j)}{\operatorname{tr}(L(x)^i\cdot L(y)^j)}$ is the Burde invariant, where $L(x)$ is the left multiplication. This invariant $c_{i,j}$ is defined as a quotient of two polynomials in the structure constants of J, for all $x,y\in J$ such that both polynomials are not zero and $c_{i,j}$ is independent of the choice of x,y.

1.4. Methods. First of all, if $J\to J'$ and $J\not\cong J'$, then $\dim\operatorname{Aut}(J)<\dim\operatorname{Aut}(J')$, where $\operatorname{Aut}(J)$ is the space of automorphisms of J. Secondly, if $J\to J'$ and $J'\to J''$, then $J\to J''$. If there is no J' such that $J\to J'$ and $J'\to J''$ are proper degenerations, then the assertion $J\to J''$ is called a *primary degeneration*. If $\dim\operatorname{Aut}(J)<\dim\operatorname{Aut}(J'')$ and there are no J' and J''' such that $J'\to J$, $J''\to J'''$, $J'\not\to J'''$ and one of the assertions $J'\to J$ and $J''\to J'''$ is a proper degeneration, then the assertion $J\not\to J''$ is called a *primary non-degeneration*. It suffices to prove only primary degenerations and non-degenerations to describe degenerations in the variety under consideration. It is easy to see that any superalgebra degenerates to the superalgebra with zero multiplication. From now on we use this fact without mentioning it.

Let us describe the methods for proving primary non-degenerations. The main tool for this is the following lemma [6].

Lemma 5. *If $J\to J'$, then the following hold:*

(1) $\dim(J^r)_i\geq\dim(J'^r)_i$, *for* $i\in\mathbb{Z}_2$;
(2) $(J)_0\to(J')_0$;
(3) $\mathfrak{a}(J)\to\mathfrak{a}(J')$;
(4) If the Burde invariant exists for J and J', then both superalgebras have the same Burde invariant;
(5) If J is associative, then J' must be associative. In fact, if J satisfies a P.I. then J' must satisfy the same P.I.

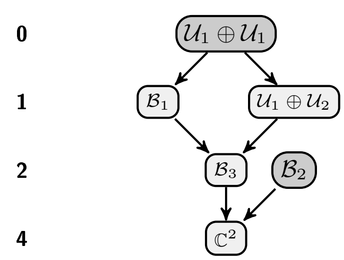
In the cases where all of these criteria can't be applied to prove $J\not\to J'$, we define $\mathcal{R}$ by a set of polynomial equations and give a basis of V, in which the structure constants of λ give a solution to all these equations. Further on, we omit the verification of the fact that $\mathcal{R}$ is stable under the action of the subgroup of upper triangular matrices and of the fact that $\mu \notin \mathcal{R}$ for any choice of a basis of V. These verifications can be done by direct calculations.

Degenerations of Graded algebras. Let G be a trivial group and let $\mathcal{V}(\mathcal{F})$ be a variety of algebras defined by a family of polynomial identities $\mathcal{F}$. It is important to notice that degeneration on the G-graded variety $G\mathcal{V}(\mathcal{F})$ is a more restrictive notion than degeneration on the variety $\mathcal{V}(\mathcal{F})$, In fact, consider $A, A' \in G\mathcal{V}(\mathcal{F})$ such that $A, A' \in \mathcal{V}(\mathcal{F})$, a degeneration between the algebras A and A' may not give rise to a degeneration between the G-graded algebras A and A', since the matrices describing the basis changes in $G\mathcal{V}(\mathcal{F})$ must preserve the G-graduation. Hence, we have the following result.

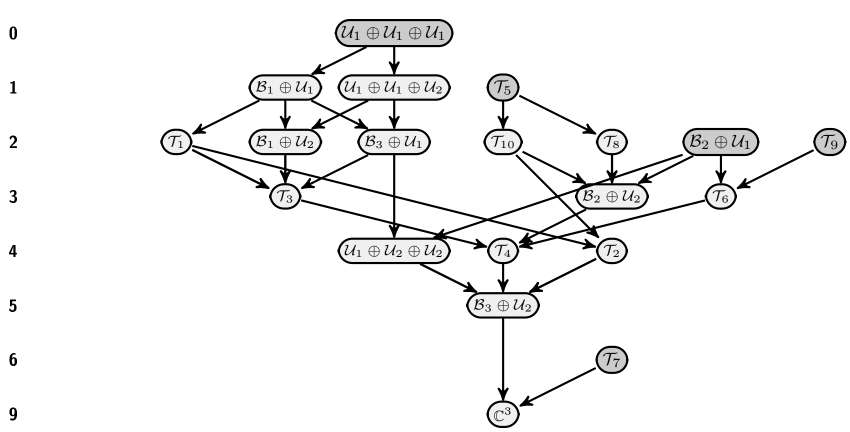
Lemma 6. *Let $A, A' \in G\mathcal{V}(\mathcal{F}) \cap \mathcal{V}(\mathcal{F})$. If $A \not\to A'$ as algebras, then $A \not\to A'$ as G-graded algebras.*

Additionally, we need the following results from [18]

Theorem 7. *The graph of primary degenerations for two-dimensional Jordan algebras has the following form:*



Theorem 8. *The graph of primary degenerations for three-dimensional Jordan algebras has the following form:*



2 Algebraic classification of four-dimensional Jordan superalgebras

All four-dimensional indecomposable Jordan superalgebras are obtained in [21] while classifying low-dimensional commutative power-associative superalgebras. Below we present our classification of all four-dimensional Jordan superalgebras.

Theorem 9. *Up to isomorphism there are 19 Jordan superalgebras of the type* $(1,3)$*, which are presented below with some additional information:*

№	*Multiplication rules*	*Decomposition*
$\mathbf{J_1}$	$f_1f_2 = e$	$\mathcal{S}_2^3 \oplus \mathcal{S}_1^1$
$\mathbf{J_2}$	$ef_1 = f_2$	$\mathcal{S}_3^3 \oplus \mathcal{S}_1^1$
$\mathbf{J_3}$	$ef_1 = f_2,\ f_1f_2 = e$	$\mathcal{S}_1^3 \oplus \mathcal{S}_1^1$
$\mathbf{J_4}$	$ef_1 = f_2,\ f_1f_3 = e$	*Indecomposable*
$\mathbf{J_5}$	$ef_1 = f_2, f_2f_3 = e$	*Indecomposable*
$\mathbf{J_6}$	$ef_1 = f_2,\ ef_2 = f_3$	*Indecomposable*
$\mathbf{J_7}$	$e^2 = e$	$\mathcal{U}_1 \oplus \mathcal{S}_1^1 \oplus \mathcal{S}_1^1 \oplus \mathcal{S}_1^1$
$\mathbf{J_8}$	$e^2 = e,\ ef_3 = \frac{1}{2}f_3$	$\mathcal{S}_1^2 \oplus \mathcal{S}_1^1 \oplus \mathcal{S}_1^1$
$\mathbf{J_9}$	$e^2 = e,\ ef_3 = f_3$	$\mathcal{S}_2^2 \oplus \mathcal{S}_1^1 \oplus \mathcal{S}_1^1$
$\mathbf{J_{10}}$	$e^2 = e,\ ef_2 = \frac{1}{2}f_2,\ ef_3 = \frac{1}{2}f_3$	$\mathcal{S}_5^3 \oplus \mathcal{S}_1^1$
$\mathbf{J_{11}}$	$e^2 = e,\ ef_2 = \frac{1}{2}f_2,\ ef_3 = \frac{1}{2}f_3,\ f_2f_3 = e$	$\mathcal{S}_7^3 \oplus \mathcal{S}_1^1$
$\mathbf{J_{12}}$	$e^2 = e,\ ef_2 = \frac{1}{2}f_2,\ ef_3 = f_3$	$\mathcal{S}_4^3 \oplus \mathcal{S}_1^1$
$\mathbf{J_{13}}$	$e^2 = e,\ ef_2 = f_2,\ ef_3 = f_3$	$\mathcal{S}_6^3 \oplus \mathcal{S}_1^1$
$\mathbf{J_{14}}$	$e^2 = e,\ ef_2 = f_2,\ ef_3 = f_3,\ f_2f_3 = e$	$\mathcal{S}_8^3 \oplus \mathcal{S}_1^1$
$\mathbf{J_{15}}$	$e^2 = e,\ ef_1 = \frac{1}{2}f_1,\ ef_2 = \frac{1}{2}f_2,\ ef_3 = \frac{1}{2}f_3$	*Indecomposable*
$\mathbf{J_{16}}$	$e^2 = e,\ ef_1 = \frac{1}{2}f_1,\ ef_2 = \frac{1}{2}f_2,\ ef_3 = f_3$	*Indecomposable*
$\mathbf{J_{17}}$	$e^2 = e,\ ef_1 = \frac{1}{2}f_1,\ ef_2 = f_2,\ ef_3 = f_3$	*Indecomposable*
$\mathbf{J_{18}}$	$e^2 = e,\ ef_1 = f_1,\ ef_2 = f_2,\ ef_3 = f_3$	*Indecomposable*
$\mathbf{J_{19}}$	$e^2 = e,\ ef_1 = f_1,\ ef_2 = f_2,\ ef_3 = f_3,\ f_1f_2 = e$	*Indecomposable*

Proof. Since $\mathfrak{J}_0$ is a Jordan algebra, we have subcases $\mathfrak{J}_0 \cong \mathcal{U}_1$ and $\mathfrak{J}_0 \cong \mathcal{U}_2$. Then we have the following multiplications for $e \in J_0,\ f_1, f_2, f_3 \in J_1$

$$\begin{aligned} ef_1 &= \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3, & f_1f_2 &= \xi_1 e, \\ ef_2 &= \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3, & f_1f_3 &= \xi_2 e, \\ ef_3 &= \gamma_1 f_1 + \gamma_2 f_2 + \gamma_3 f_3, & f_2f_3 &= \xi_3 e. \end{aligned}$$

The linear operator $L_x : \mathfrak{J} \to \mathfrak{J}, x \in \mathfrak{J}$ such that $L_x(y) = xy$ is called a left multiplication operator. It is obvious that:

$$x \in \mathfrak{J}_0,\ L_x : \mathfrak{J}_0 \to \mathfrak{J}_0,\ L_x : \mathfrak{J}_1 \to \mathfrak{J}_1,$$
$$x \in \mathfrak{J}_1,\ L_x : \mathfrak{J}_0 \to \mathfrak{J}_1,\ L_x : \mathfrak{J}_1 \to \mathfrak{J}_0.$$

For the action of the operator L_e on $\mathfrak{J}_1$ we can write the following matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}.$$

However, it is easy to prove that, by using a simple change of basis, the matrix of L_e has one of the following forms:

$$\begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \quad \begin{pmatrix} \mu_1 & 1 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \quad \begin{pmatrix} \mu_1 & 1 & 0 \\ 0 & \mu_1 & 1 \\ 0 & 0 & \mu_1 \end{pmatrix}.$$

<u>Let $\mathfrak{J}_0 \cong \mathcal{U}_2$.</u>

I) Let $L_e \simeq \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$, then the rule of multiplication can be written as follows:

$$ef_1 = \mu_1 f_1, \quad ef_2 = \mu_2 f_2, \quad ef_3 = \mu_3 f_3,$$
$$f_1 f_2 = \xi_1 e, \quad f_1 f_3 = \xi_2 e, \quad f_2 f_3 = \xi_3 e.$$

In this case, from $J(e, e, e, f_1) = 0$, $J(e, e, e, f_2) = 0$, and $J(e, e, e, f_3) = 0$ we get $\mu_i = 0,\ i = \overline{1, 3}$.

If $(\xi_1, \xi_2, \xi_3) \neq (0, 0, 0)$, then by changing $e' = \xi_1 e,\ f_3' = \xi_1 f_3 - \xi_2 f_2 + \xi_3 f_1$ we obtian the Jordan superalgebra:

$$e'^2 = 0,\ f_1 f_2 = e'.$$

We denote this superalgebra by $\mathbf{J_1}$.

II) Let $L_e \simeq \begin{pmatrix} \mu_1 & 1 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$, then the rule of multiplication can be written as follows:

$$ef_1 = \mu_1 f_1 + f_2, \quad ef_2 = \mu_1 f_2, \quad ef_3 = \mu_3 f_3,$$
$$f_1 f_2 = \xi_1 e, \quad f_1 f_3 = \xi_2 e, \quad f_2 f_3 = \xi_3 e.$$

In this case, from $J(e, e, e, f_2) = 0$ and $J(e, e, e, f_3) = 0$ we get $2\mu_1^3 = 0,\ 2\mu_3^3 = 0$, respectively. So $\mu_1 = \mu_3 = 0$. When $(\xi_1, \xi_2, \xi_3) = (0, 0, 0)$ we get the superalgebra

$$ef_1 = f_2,$$

which we denote by $\mathbf{J_2}$. Hence, if $(\xi_1, \xi_2, \xi_3) \neq (0,0,0)$, then we can change the basis as follows:

$$\begin{array}{lll} f'_1 = a_1 f_1 + a_2 f_2 + a_3 f_3, & f'_2 = a_1 f_2, & f'_3 = b_1 f_2 + b_2 f_3, \\ f'_1 f'_2 = a_1^2 \xi_1 e - a_1 a_3 \xi_3 e = \xi'_1 e, & & f'_2 f'_3 = a_1 b_2 \xi_3 e. \\ \multicolumn{3}{l}{f'_1 f'_3 = a_1 b_1 \xi_1 e + a_1 b_2 \xi_2 e + a_2 b_2 \xi_3 e - a_3 b_1 \xi_3 e = \xi'_2 e,} \end{array}$$

We proceed as follows:

a) If $\xi_3 = 0$ then $\xi'_1 = a_1^2 \xi_1,\ \xi'_2 = a_1 b_1 \xi_1 + a_1 b_2 \xi_2$.

a.1) If $\xi_1 \neq 0$ then by choosing $b_1 = -\frac{b_2 \xi_2}{\xi_1},\ a_1 = \frac{1}{\xi_1}$ we get $\xi'_2 = 0$ and $\xi'_1 = 1$ which gives the superalgebra $\mathbf{J_3}$:

$$e^2 = 0,\ e f'_1 = f'_2,\ f'_1 f'_2 = e.$$

a.2) If $\xi_1 = 0$ then $\xi'_2 = a_1 b_2 \xi_2$. In this case by choosing $a_1 b_2 = \frac{1}{\xi_2}$ we get the superalgebra $\mathbf{J_4}$:

$$e^2 = 0,\ e f'_1 = f'_2,\ f'_1 f'_3 = e.$$

b) If $\xi_3 \neq 0$ then by choosing $a_3 = \frac{a_1 \xi_1}{\xi_3},\ b_2 = \frac{1}{a_1 \xi_3},\ a_2 = -\frac{a_1 \xi_2}{\xi_3}$ and taking $b_2 = 1$ for simplicity we get the superalgebra $\mathbf{J_5}$: $e^2 = 0,\ e f'_1 = f'_2,\ f'_2 f'_3 = e$.

III) Let $L_e \simeq \begin{pmatrix} \mu_1 & 1 & 0 \\ 0 & \mu_1 & 1 \\ 0 & 0 & \mu_1 \end{pmatrix}$, then the rule of multiplication can be written as follows:

$$\begin{array}{lll} e f_1 = \mu_1 f_1 + f_2, & e f_2 = \mu_1 f_2 + f_3, & e f_3 = \mu_1 f_3, \\ f_1 f_2 = \xi_1 e, & f_1 f_3 = \xi_2 e, & f_2 f_3 = \xi_3 e. \end{array}$$

In this case from $J(e,e,e,f_1) = 0$ we get $\mu_1 = 0$. Moreover, from $J(e,e,f_1,f_2) = 0$ and $J(e,f_1,f_1,f_2) = 0$ we get $\xi_3 = 0$ and $\xi_2 = \xi_1 = 0$, respectively. As a result, we have the superalgebra $\mathbf{J_6}$:

$$e f_1 = f_2,\ e f_2 = f_3.$$

Let $\mathfrak{J}_0 \cong \mathcal{U}_1$.

I) Let $L_e \simeq \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$, then the rule of multiplication can be written as follows:

$$\begin{array}{lll} e f_1 = \mu_1 f_1, & e f_2 = \mu_2 f_2, & e f_3 = \mu_3 f_3, \\ f_1 f_2 = \xi_1 e, & f_1 f_3 = \xi_2 e, & f_2 f_3 = \xi_3 e. \end{array}$$

From $J(e,e,e,f_1) = 0$, $J(e,e,e,f_2) = 0$, and $J(e,e,e,f_3) = 0$ we obtain equations below

$$(\mu_i - 1)\mu_i(2\mu_i - 1) = 0, \quad i = \overline{1,3}.$$

Up to permutation of f_1, f_2 and f_3 we have ten possibilities:
$(\mu_1, \mu_2, \mu_3) \in \{(0,0,0), (0,0,\frac{1}{2}), (0,0,1), (0,\frac{1}{2},\frac{1}{2}), (0,\frac{1}{2},1), (0,1,1), (\frac{1}{2},\frac{1}{2},\frac{1}{2}), (\frac{1}{2},\frac{1}{2},1), (\frac{1}{2},1,1), (1,1,1)\}$.

1. $(\mu_1, \mu_2, \mu_3) = (0,0,0)$. In this case, we have the following results
$$J(e,e,f_1,f_2) = 0 \Rightarrow \xi_1 = 0, \quad J(e,e,f_1,f_3) = 0 \Rightarrow \xi_2 = 0,$$
$$J(e,e,f_2,f_3) = 0 \Rightarrow \xi_3 = 0.$$
Hence, the obtained superalgebra is $\mathbf{J_7}$.
2. $(\mu_1, \mu_2, \mu_3) = (0,0,\frac{1}{2})$. In this case, we have the following results
$$J(e,e,f_1,f_2) = 0 \Rightarrow \xi_1 = 0, \quad J(e,e,f_3,f_1) = 0 \Rightarrow \xi_2 = 0,$$
$$J(e,e,f_3,f_2) = 0 \Rightarrow \xi_3 = 0.$$
Hence, the the obtained superalgebra is $\mathbf{J_8}$.
3. $(\mu_1, \mu_2, \mu_3) = (0,0,1)$. In this case, we have the following results
$$J(e,e,f_1,f_2) = 0 \Rightarrow \xi_1 = 0, \quad J(e,e,f_1,f_3) = 0 \Rightarrow \xi_2 = 0,$$
$$J(e,e,f_2,f_3) = 0 \Rightarrow \xi_3 = 0.$$
Hence, the the obtained superalgebra is $\mathbf{J_9}$.
4. $(\mu_1, \mu_2, \mu_3) = (0,\frac{1}{2},\frac{1}{2})$. In this case, we have the following results
$$J(e,e,f_1,f_2) = 0 \Rightarrow \xi_1 = 0, \quad J(e,e,f_1,f_3) = 0 \Rightarrow \xi_2 = 0.$$
Hence, the obtained superalgebras are $\mathbf{J_{10}}$ and $\mathbf{J_{11}}$.
5. $(\mu_1, \mu_2, \mu_3) = (0,\frac{1}{2},1)$ In this case, we have the following results
$$J(e,e,f_2,f_1) = 0 \Rightarrow \xi_1 = 0, \quad J(e,e,f_1,f_3) = 0 \Rightarrow \xi_2 = 0,$$
$$J(e,e,f_2,f_3) = 0 \Rightarrow \xi_3 = 0.$$
Hence, the obtained superalgebra is $\mathbf{J_{12}}$.
6. $(\mu_1, \mu_2, \mu_3) = (0,1,1)$. In this case, we have the following results
$$J(e,e,f_1,f_2) = 0 \Rightarrow \xi_1 = 0, \quad J(e,e,f_1,f_3) = 0 \Rightarrow \xi_2 = 0.$$
Hence, the obtained superalgebras are $\mathbf{J_{13}}$ and $\mathbf{J_{14}}$.
7. $(\mu_1, \mu_2, \mu_3) = (0,\frac{1}{2},\frac{1}{2})$. In this case, we have the following results
$$J(f_1,f_2,e,f_3) = 0 \Rightarrow \xi_1 = \xi_2 = \xi_3 = 0.$$
Hence, the obtained superalgebra is $\mathbf{J_{15}}$.
8. $(\mu_1, \mu_2, \mu_3) = (\frac{1}{2},\frac{1}{2},1)$. In this case, we have the following results
$$J(e,f_1,f_2,f_3) = 0 \Rightarrow \xi_1 = \xi_3 = 0, \quad J(e,e,f_1,f_3) = 0 \Rightarrow \xi_2 = 0.$$
Hence, the obtained superalgebra is $\mathbf{J_{16}}$.
9. $(\mu_1, \mu_2, \mu_3) = (\frac{1}{2},1,1)$. In this case, we have the following results
$$J(e,f_1,f_2,f_3) = 0 \Rightarrow \xi_1 = \xi_3 = 0, \quad J(e,e,f_1,f_3) = 0 \Rightarrow \xi_2 = 0.$$
Hence, the obtained superalgebra is $\mathbf{J_{17}}$.
10. $(\mu_1, \mu_2, \mu_3) = (1,1,1)$. In this case, the multiplication rules in the obtained superalgebra are $e^2 = e, \quad ef_1 = f_1, \; ef_2 = f_2, \; ef_3 =$

$f_3,\ f_1f_2=\xi_1e,\ f_1f_3=\xi_2e,\ f_2f_3=\xi_3e$. When $(\xi_1,\xi_2,\xi_3)=(0,0,0)$ we get the superalgebra $\mathbf{J_{18}}$.

However, when $(\xi_1,\xi_2,\xi_3)\neq(0,0,0)$ we can assume that $\xi_1\neq 0$ and by changing the basis as follows

$$f_1'=\tfrac{1}{\xi_1}f_1+f_2,\quad f_2'=f_2,\quad f_3'=\xi_1f_3-\xi_2f_2+\xi_3f_1,$$

we get the superalgebra $\mathbf{J_{19}}$.

II) Let $L_e\simeq\begin{pmatrix}\mu_1 & 1 & 0\\ 0 & \mu_1 & 0\\ 0 & 0 & \mu_3\end{pmatrix}$, then the rule of multiplication can be written as follows:

$$\begin{array}{lll} ef_1=\mu_1f_1+f_2, & ef_2=\mu_1f_2, & ef_3=\mu_3f_3,\\ f_1f_2=\xi_1e, & f_1f_3=\xi_2e, & f_2f_3=\xi_3e.\end{array}$$

However, the following two equations

$$\begin{array}{l} J(e,e,e,f_1)=0\ \Rightarrow\ 1-6\mu_1+6\mu_1^2=0,\\ J(e,e,e,f_2)=0\ \Rightarrow\ (\mu_1-1)\mu_1(2\mu_1-1)=0,\end{array}$$

which can not be satisfied at the same time, give a contradiction, thereby no superalgebras can be found in this subcase.

III) Let $L_e\simeq\begin{pmatrix}\mu_1 & 1 & 0\\ 0 & \mu_1 & 1\\ 0 & 0 & \mu_1\end{pmatrix}$, then the rule of multiplication can be written as follows:

$$\begin{array}{lll} ef_1=\mu_1f_1+f_2, & ef_2=\mu_1f_2+f_3, & ef_3=\mu_1f_3,\\ f_1f_2=\xi_1e, & f_1f_3=\xi_2e, & f_2f_3=\xi_3e.\end{array}$$

However, the following two equations

$$\begin{array}{l} J(e,e,e,f_2)=0\ \Rightarrow\ 1-6\mu_1+6\mu_1^2=0,\\ J(e,e,e,f_3)=0\ \Rightarrow\ (\mu_1-1)\mu_1(2\mu_1-1)=0,\end{array}$$

which can not be satisfied at the same time, give a contradiction, thereby no superalgebras can be found in this subcase. □

Theorem 10. *Up to isomorphism there are 71 Jordan superalgebras of type $(2,2)$, which are presented below with some additional information:*

№	*Multiplication rules*	*Decomposition*
$\mathcal{J}_1$	$e_1^2=e_1,\ e_2^2=e_2$	$\mathcal{U}_1\oplus\mathcal{U}_1\oplus\mathcal{S}_1^1\oplus\mathcal{S}_1^1$
$\mathcal{J}_2$	$e_1^2=e_1,\ e_2^2=e_2,\ e_2f_2=f_2$	$\mathcal{U}_1\oplus\mathcal{S}_2^2\oplus S_1^1$
$\mathcal{J}_3$	$e_1^2=e_1,\ e_2^2=e_2,\ e_2f_2=\frac{1}{2}f_2$	$\mathcal{U}_1\oplus\mathcal{S}_1^2\oplus S_1^1$
$\mathcal{J}_4$	$e_1^2=e_1,\ e_2^2=e_2,\ e_2f_1=f_1,\ e_2f_2=f_2$	$\mathcal{U}_1\oplus\mathcal{S}_6^3$
$\mathcal{J}_5$	$e_1^2=e_1,\ e_2^2=e_2,\ e_2f_1=f_1,\ e_2f_2=\frac{1}{2}f_2$	$\mathcal{U}_1\oplus\mathcal{S}_4^3$
$\mathcal{J}_6$	$e_1^2=e_1,\ e_2^2=e_2,\ e_2f_1=f_1,\ e_2f_2=f_2,\ f_1f_2=e_2$	$\mathcal{U}_1\oplus\mathcal{S}_8^3$
$\mathcal{J}_7$	$e_1^2=e_1,\ e_2^2=e_2,\ e_2f_1=\frac{1}{2}f_1,\ e_2f_2=\frac{1}{2}f_2$	$\mathcal{U}_1\oplus\mathcal{S}_5^3$

$\mathcal{J}_{\mathbf{8}}$	$e_1^2=e_1,\ e_2^2=e_2,\ e_2f_1=\frac{1}{2}f_1,\ e_2f_2=\frac{1}{2}f_2,\ \ f_1f_2=e_2$	$\mathcal{U}_1\oplus\mathcal{S}_7^3$
$\mathcal{J}_{\mathbf{9}}$	$e_1^2=e_1,\ e_2^2=e_2,\ e_1f_2=\frac{1}{2}f_2,\ e_2f_2=\frac{1}{2}f_2$	$\mathcal{S}_{13}^3\oplus\mathcal{S}_1^1$
$\mathcal{J}_{\mathbf{10}}$	$e_1^2=e_1,\ e_2^2=e_2,\ e_1f_2=\frac{1}{2}f_2,\ e_2f_1=f_1$	$\mathcal{S}_1^2\oplus\mathcal{S}_2^2$
$\mathcal{J}_{\mathbf{11}}$	$e_1^2=e_1,\ e_2^2=e_2,\ e_1f_2=\frac{1}{2}f_2,\ e_2f_1=f_1,\ e_2f_2=\frac{1}{2}f_2$	*Indecomposable*
$\mathcal{J}_{\mathbf{12}}$	$e_1^2=e_1,\ e_2^2=e_2,\ e_1f_2=\frac{1}{2}f_2,\ e_2f_1=\frac{1}{2}f_1$	$\mathcal{S}_1^2\oplus\mathcal{S}_1^2$
$\mathcal{J}_{\mathbf{13}}$	$e_1^2=e_1,\ e_2^2=e_2,\ e_1f_2=\frac{1}{2}f_2,\ e_2f_1=\frac{1}{2}f_1,\ e_2f_2=\frac{1}{2}f_2$	*Indecomposable*
$\mathcal{J}_{\mathbf{14}}$	$e_1^2=e_1,\ e_2^2=e_2,\ e_1f_2=f_2,\ e_2f_1=f_1$	$\mathcal{S}_2^2\oplus\mathcal{S}_2^2$
$\mathcal{J}_{\mathbf{15}}$	$e_1^2=e_1,\ e_2^2=e_2,\ e_1f_1=\frac{1}{2}f_1,$ $e_1f_2=\frac{1}{2}f_2,e_2f_1=\frac{1}{2}f_1,\ e_2f_2=\frac{1}{2}f_2$	*Indecomposable*
$\mathcal{J}_{\mathbf{16}}^{\mathbf{t}}$	$e_1^2=e_1,\ e_2^2=e_2,\ e_1f_1=\frac{1}{2}f_1,\ e_1f_2=\frac{1}{2}f_2,$ $e_2f_1=\frac{1}{2}f_1,\ e_2f_2=\frac{1}{2}f_2,\ f_1f_2=e_1+te_2$	*Indecomposable*
$\mathcal{J}_{\mathbf{17}}$	$f_1f_2=e_1$	$\mathcal{U}_2\oplus\mathcal{S}_2^3$
$\mathcal{J}_{\mathbf{18}}$	$e_2f_1=f_2$	$\mathcal{U}_2\oplus\mathcal{S}_3^3$
$\mathcal{J}_{\mathbf{19}}$	$e_2f_1=f_2,\ f_1f_2=e_1$	*Indecomposable*
$\mathcal{J}_{\mathbf{20}}$	$e_2f_1=f_2,\ f_1f_2=e_2$	$\mathcal{U}_2\oplus\mathcal{S}_1^3$
$\mathcal{J}_{\mathbf{21}}$	$e_1^2=e_1$	$\mathcal{U}_1\oplus\mathcal{U}_2\oplus\mathcal{S}_1^1\oplus\mathcal{S}_1^1$
$\mathcal{J}_{\mathbf{22}}$	$e_1^2=e_1,\ f_1f_2=e_2$	$\mathcal{U}_1\oplus\mathcal{S}_2^3$
$\mathcal{J}_{\mathbf{23}}$	$e_1^2=e_1,\ e_2f_1=f_2$	$\mathcal{U}_1\oplus\mathcal{S}_3^3$
$\mathcal{J}_{\mathbf{24}}$	$e_1^2=e_1,\ e_2f_1=f_2,\ f_1f_2=e_2$	$\mathcal{U}_1\oplus\mathcal{S}_1^3$
$\mathcal{J}_{\mathbf{25}}$	$e_1^2=e_1,\ e_1f_2=\frac{1}{2}f_2$	$\mathcal{U}_2\oplus\mathcal{S}_1^2\oplus\mathcal{S}_1^1$
$\mathcal{J}_{\mathbf{26}}$	$e_1^2=e_1,\ e_1f_2=f_2$	$\mathcal{U}_2\oplus\mathcal{S}_2^2\oplus\mathcal{S}_1^1$
$\mathcal{J}_{\mathbf{27}}$	$e_1^2=e_1,\ e_1f_1=\frac{1}{2}f_1,\ e_1f_2=\frac{1}{2}f_2$	$\mathcal{U}_2\oplus\mathcal{S}_5^3$
$\mathcal{J}_{\mathbf{28}}$	$e_1^2=e_1,\ e_1f_1=\frac{1}{2}f_1,\ e_1f_2=\frac{1}{2}f_2,\ f_1f_2=e_2$	*Indecomposable*
$\mathcal{J}_{\mathbf{29}}$	$e_1^2=e_1,\ e_1f_1=\frac{1}{2}f_1,\ e_1f_2=\frac{1}{2}f_2,\ f_1f_2=e_1$	$\mathcal{U}_2\oplus\mathcal{S}_7^3$
$\mathcal{J}_{\mathbf{30}}$	$e_1^2=e_1,\ e_1f_1=\frac{1}{2}f_1,\ e_1f_2=\frac{1}{2}f_2,\ f_1f_2=e_1+e_2$	*Indecomposable*
$\mathcal{J}_{\mathbf{31}}$	$e_1^2=e_1,\ e_1f_1=\frac{1}{2}f_1,\ e_1f_2=\frac{1}{2}f_2,\ e_2f_1=f_2$	*Indecomposable*
$\mathcal{J}_{\mathbf{32}}$	$e_1^2=e_1,\ e_1f_1=\frac{1}{2}f_1,\ e_1f_2=\frac{1}{2}f_2,\ e_2f_1=f_2,\ f_1f_2=e_2$	*Indecomposable*
$\mathcal{J}_{\mathbf{33}}$	$e_1^2=e_1,\ e_1f_1=\frac{1}{2}f_1,\ e_1f_2=f_2$	$\mathcal{U}_2\oplus\mathcal{S}_4^3$
$\mathcal{J}_{\mathbf{34}}$	$e_1^2=e_1,\ e_1f_1=f_1,\ e_1f_2=f_2$	$\mathcal{U}_2\oplus\mathcal{S}_6^3$
$\mathcal{J}_{\mathbf{35}}$	$e_1^2=e_1,\ e_1f_1=f_1,\ e_1f_2=f_2,\ f_1f_2=e_1$	$\mathcal{U}_2\oplus\mathcal{S}_8^3$
$\mathcal{J}_{\mathbf{36}}$	$e_1^2=e_1,\ e_1e_2=e_2$	$\mathcal{B}_1\oplus\mathcal{S}_1^1\oplus\mathcal{S}_1^1$
$\mathcal{J}_{\mathbf{37}}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_1f_2=\frac{1}{2}f_2$	$\mathcal{S}_9^3\oplus\mathcal{S}_1^1$
$\mathcal{J}_{\mathbf{38}}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_1f_2=f_2$	$\mathcal{S}_{10}^3\oplus\mathcal{S}_1^1$
$\mathcal{J}_{\mathbf{39}}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_1f_1=\frac{1}{2}f_1,\ e_1f_2=\frac{1}{2}f_2$	*Indecomposable*
$\mathcal{J}_{\mathbf{40}}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_1f_1=\frac{1}{2}f_1,\ e_1f_2=\frac{1}{2}f_2,\ f_1f_2=e_2$	*Indecomposable*
$\mathcal{J}_{\mathbf{41}}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_1f_1=\frac{1}{2}f_1,\ e_1f_2=\frac{1}{2}f_2,\ e_2f_1=f_2$	*Indecomposable*
$\mathcal{J}_{\mathbf{42}}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_1f_1=\frac{1}{2}f_1,$ $e_1f_2=\frac{1}{2}f_2,\ e_2f_1=f_2,\ f_1f_2=e_2$	*Indecomposable*
$\mathcal{J}_{\mathbf{43}}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_1f_1=\frac{1}{2}f_1,\ e_1f_2=f_2$	*Indecomposable*
$\mathcal{J}_{\mathbf{44}}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_1f_1=f_1,\ e_1f_2=f_2$	*Indecomposable*
$\mathcal{J}_{\mathbf{45}}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_1f_1=f_1,\ e_1f_2=f_2,\ f_1f_2=e_2$	*Indecomposable*
$\mathcal{J}_{\mathbf{46}}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_1f_1=f_1,\ e_1f_2=f_2,\ f_1f_2=e_1$	*Indecomposable*
$\mathcal{J}_{\mathbf{47}}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_1f_1=f_1,\ e_1f_2=f_2,\ f_1f_2=e_1+e_2$	*Indecomposable*
$\mathcal{J}_{\mathbf{48}}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_1f_1=f_1,\ e_1f_2=f_2,\ e_2f_1=f_2$	*Indecomposable*
$\mathcal{J}_{\mathbf{49}}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_1f_1=f_1,\ e_1f_2=f_2,\ e_2f_1=f_2,\ f_1f_2=e_2$	*Indecomposable*
$\mathcal{J}_{\mathbf{50}}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2$	$\mathcal{B}_2\oplus\mathcal{S}_1^1\oplus\mathcal{S}_1^1$
$\mathcal{J}_{\mathbf{51}}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_1f_2=\frac{1}{2}f_2$	$\mathcal{S}_{11}^3\oplus\mathcal{S}_1^1$
$\mathcal{J}_{\mathbf{52}}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_1f_2=\frac{1}{2}f_2,\ f_1f_2=e_2$	*Indecomposable*

$\mathcal{J}_{53}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_1f_2=\frac{1}{2}f_2,\ e_2f_2=f_1$	*Indecomposable*
$\mathcal{J}_{54}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_1f_2=\frac{1}{2}f_2,\ e_2f_2=f_1,\ f_1f_2=e_2$	*Indecomposable*
$\mathcal{J}_{55}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_1f_2=\frac{1}{2}f_2,\ e_2f_1=f_2$	*Indecomposable*
$\mathcal{J}_{56}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_1f_2=\frac{1}{2}f_2,\ e_2f_1=f_2,\ f_1f_2=e_2$	*Indecomposable*
$\mathcal{J}_{57}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_1f_2=f_2$	$\mathcal{S}_{12}^3\oplus\mathcal{S}_1^1$
$\mathcal{J}_{58}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_1f_1=\frac{1}{2}f_1,\ e_1f_2=\frac{1}{2}f_2$	*Indecomposable*
$\mathcal{J}_{59}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_1f_1=\frac{1}{2}f_1,\ e_1f_2=f_2$	*Indecomposable*
$\mathcal{J}_{60}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_1f_1=\frac{1}{2}f_1,\ e_1f_2=f_2,\ f_1f_2=e_2$	*Indecomposable*
$\mathcal{J}_{61}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_1f_1=\frac{1}{2}f_1,\ e_1f_2=f_2,\ e_2f_2=f_1$	*Indecomposable*
$\mathcal{J}_{62}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_1f_1=\frac{1}{2}f_1,$ $e_1f_2=f_2,\ e_2f_2=f_1,\ f_1f_2=e_2$	*Indecomposable*
$\mathcal{J}_{63}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_1f_1=\frac{1}{2}f_1,\ e_1f_2=f_2,\ e_2f_1=f_2$	*Indecomposable*
$\mathcal{J}_{64}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_1f_1=\frac{1}{2}f_1,$ $e_1f_2=f_2,\ e_2f_1=f_2,\ f_1f_2=e_2$	*Indecomposable*
$\mathcal{J}_{65}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_1f_1=f_1,\ e_1f_2=f_2$	*Indecomposable*
$\mathcal{J}_{66}$	$e_1^2=e_2$	$\mathcal{B}_3\oplus\mathcal{S}_1^1\oplus\mathcal{S}_1^1$
$\mathcal{J}_{67}$	$e_1^2=e_2,\ f_1f_2=e_2$	*Indecomposable*
$\mathcal{J}_{68}$	$e_1^2=e_2,\ f_1f_2=e_1$	*Indecomposable*
$\mathcal{J}_{69}$	$e_1^2=e_2,\ e_2f_1=f_2$	*Indecomposable*
$\mathcal{J}_{70}$	$e_1^2=e_2,\ e_1f_1=f_2$	*Indecomposable*
$\mathcal{J}_{71}$	$e_1^2=e_2,\ e_1f_1=f_2,\ f_1f_2=e_2$	*Indecomposable*

Proof. Let $\mathfrak{J}_0\cong\mathcal{U}_1\oplus\mathcal{U}_1$. Here we are looking for Jordan superalgebras such that $\mathfrak{J}=(\mathbb{F}e_1+\mathbb{F}e_2)+(\mathbb{F}f_1+\mathbb{F}f_2)$ with multiplication rules

$$e_1^2=e_1,\ \ e_2^2=e_2,\ \ e_1f_1=\alpha_1f_1+\alpha_2f_2,\ \ e_1f_2=\alpha_3f_1+\alpha_4f_2,$$
$$e_2f_1=\beta_1f_1+\beta_2f_2,\ \ f_1f_2=\xi_1e_1+\xi_2e_2,\ \ e_2f_2=\beta_3f_1+\beta_4f_2.$$

For the action of the operator L_{e_1} on $\mathfrak{J}_1$ we can write the following matrix

$$\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{pmatrix}.$$

However, it is easy to prove that, by using a simple change of basis, the matrix of L_{e_1} have one of the following forms:

$$\begin{pmatrix}\mu_1 & 0\\ 0 & \mu_2\end{pmatrix},\quad \begin{pmatrix}\mu_1 & 1\\ 0 & \mu_1\end{pmatrix}.$$

I) Let $L_{e_1}\simeq\begin{pmatrix}\mu_1 & 0\\ 0 & \mu_2\end{pmatrix}$ then the rule of multiplication can be written as follows:

$$e_1^2=e_1,\ \ e_2^2=e_2,\ \ e_1f_1=\mu_1f_1,\ \ e_1f_2=\mu_2f_2,$$
$$e_2f_1=\beta_1f_1+\beta_2f_2,\ \ f_1f_2=\xi_1e_1+\xi_2e_2,\ \ e_2f_2=\beta_3f_1+\beta_4f_2.$$

From $J(e_1,e_1,e_1,f_1)=0$, $J(e_1,e_1,e_1,f_2)=0$, we obtain equations below

$$(\mu_i - 1)\mu_i(2\mu_i - 1) = 0, \quad i = \overline{1,2}.$$

Up to permutation of f_1 and f_2 we have six possibilities:

$$(\mu_1, \mu_2) \in \{(0,0), (0, \tfrac{1}{2}), (0,1), (\tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{1}{2}, 1), (1,1)\}.$$

1. $(\mu_1, \mu_2) = (0,0)$ In this case, we can consider the action of e_2. By using a simple change of basis, the matrix of L_{e_2} have one of the following forms:

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}.$$

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$ we have

$$\xi_1 = 0, \qquad (\tau_1 - 1)\tau_1(2\tau_1 - 1) = 0, \; (\tau_2 - 1)\tau_2(2\tau_2 - 1) = 0$$

from $J(e_1, e_1, f_1, f_2) = 0$, $J(e_2, e_2, e_2, f_1) = 0$ and $J(e_2, e_2, e_2, f_2) = 0$, respectively.

- If $\tau_1 = 0$ then from
$$J(e_2, e_2, f_1, f_2) = 0 \;\Rightarrow\; \xi_2(2\tau_2 - 1) = 0,$$
$$J(e_2, e_2, f_2, f_1) = 0 \;\Rightarrow\; \xi_2(\tau_2 - 1) = 0.$$
we get $\xi_2 = 0$ and $\tau_2 \in 0, 1, \frac{1}{2}$, which gives superalgebras $\mathcal{J}_1, \mathcal{J}_2$ and $\mathcal{J}_3$.
- If $\tau_1 = 1$ then we have
$$J(e_2, e_2, f_2, f_1) = 0 \;\Rightarrow\; \xi_2(\tau_2 - 1) = 0.$$
So, either $\xi_2 = 0$ and we have Jordan superalgebras $\mathcal{J}_4$ and $\mathcal{J}_5$ with $\tau_2 \in \{1, \frac{1}{2}\}$, or $\xi_2 \neq 0$ and $\tau_2 = 1$ which gives us the Jordan superalgebra $\mathcal{J}_6$.
- If $\tau_1 = \frac{1}{2}$ then from
$$J(e_2, e_2, f_1, f_2) = 0 \;\Rightarrow\; \xi_2(2\tau_2 - 1) = 0.$$
So, either $\xi_2 = 0$ and we have a Jordan superalgebra $\mathcal{J}_7$ with $\tau_2 = \frac{1}{2}$ ($\tau_2 \in \{0,1\}$ repeat the previous superalgebras), or $\xi_2 \neq 0$ and $\tau_2 = \frac{1}{2}$ which gives us the Jordan superalgebra $\mathcal{J}_8$.

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}$ we have:

$$J(e_2, e_2, e_2, f_1) = 0 \;\Rightarrow\; (1 - 6\tau_1 + 6\tau_1^2) = 0,$$
$$J(e_2, e_2, e_2, f_2) = 0 \;\Rightarrow\; (\tau_1 - 1)\tau_1(2\tau_1 - 1) = 0.$$

This contradiction implies that no superalgebras can be found in this case.

2. $(\mu_1, \mu_2) = (0, \frac{1}{2})$. In this case we have the following results

$$J(e_1, e_1, e_2, f_1) = 0 \;\Rightarrow\; \beta_2 = 0, \;\; J(e_1, e_1, f_2, e_2) = 0 \;\Rightarrow\; \beta_3 = 0,$$
$$J(e_1, e_1, f_2, f_1) = 0 \;\Rightarrow\; \xi_1 = \xi_2 = 0,$$
$$J(e_1, e_2, e_2, f_2) = 0 \;\Rightarrow\; \beta_4(2\beta_4 - 1) = 0,$$
$$J(e_2, e_2, e_2, f_1) = 0 \;\Rightarrow\; (\beta_1 - 1)\beta_1(2\beta_1 - 1) = 0.$$

Hence, in this case, we obtain 6 Jordan superalgebras with $\beta_4 \in \{0, \frac{1}{2}\}$ and $\beta_1 \in \{0, 1, \frac{1}{2}\}$. Among them we have $\mathcal{J}_9, \mathcal{J}_{10}, \mathcal{J}_{11}, \mathcal{J}_{12}, \mathcal{J}_{13}$, while $\beta_1 = \beta_4 = 0$ gives a superalgebra isomorphic to previously obtained one.

3. $(\mu_1, \mu_2) = (0, 1)$ In this case we have the following results

$$\begin{gathered} J(e_1, e_1, e_2, f_1) = 0 \ \Rightarrow \ \beta_2 = 0, \\ J(e_1, e_1, e_2, f_2) = 0 \ \Rightarrow \ \beta_3 = \beta_4 = 0, \\ J(e_1, e_1, f_1, f_2) = 0 \ \Rightarrow \ \xi_1 = 0, \ \ J(e_1, e_1, f_2, f_1) = 0 \ \Rightarrow \ \xi_2 = 0, \\ J(e_2, e_2, e_2, f_1) = 0 \ \Rightarrow \ (\beta_1 - 1)\beta_1(2\beta_1 - 1) = 0. \end{gathered}$$

Hence, in this case, we obtain 3 Jordan superalgebras with $\beta_1 \in \{0, 1, \frac{1}{2}\}$. Only $\beta_1 = 1$ gives a new superalgebra $\mathcal{J}_{14}$.

4. $(\mu_1, \mu_2) = (\frac{1}{2}, \frac{1}{2})$. In this case, we can consider the action of e_2. By using a simple change of basis, the matrix of L_{e_2} has one of the following forms:

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}.$$

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$ we have:

$$\begin{gathered} J(e_1, e_2, e_2, f_1) = 0 \ \Rightarrow \ \tau_1(2\tau_1 - 1) = 0, \\ J(e_1, e_2, e_2, f_4) = 0 \ \Rightarrow \ \tau_2(2\tau_2 - 1) = 0. \end{gathered}$$

- If $(\tau_1, \tau_2) = (\frac{1}{2}, \frac{1}{2})$ then we get the well-known one parametric family of four-dimensional Jordan superalgebras, which we denoted as $\mathcal{J}_{16}^t$.
- If $(\tau_1, \tau_2) = (0, \frac{1}{2})$ then from $J(e_1, e_2, f_1, f_2) = 0$ we get $\xi_1 = \xi_2 = 0$ which gives a Jordan superalgebra isomorphic to $\mathcal{J}_{13}$.
- If $(\tau_1, \tau_2) = (0, 0)$ then from $J(e_1, e_2, f_1, f_2) = 0$ we get $\xi_2 = 0$ and hence the following Jordan superalgebra: $e_1^2 = e_1, \ e_2^2 = e_2, \ e_1 f_1 = \frac{1}{2} f_1, \ e_1 f_2 = \frac{1}{2} f_2, \ f_1 f_2 = \xi_1 e_1$. Though only $\xi_1 = 0$ gives us a new superalgebra $\mathcal{J}_{15}$.

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}$ we have:

$$\begin{gathered} J(e_1, e_2, e_2, f_1) = 0 \ \Rightarrow \ 2\tau_1 - \tfrac{1}{2} = 0, \\ J(e_1, e_2, e_2, f_4) = 0 \ \Rightarrow \ \tau_1(2\tau_1 - 1) = 0. \end{gathered}$$

This is a contradiction.

5. $(\mu_1, \mu_2) = (\frac{1}{2}, 1)$. In this case we have the following results

$$\begin{gathered} J(e_1, e_1, e_2, f_2) = 0 \ \Rightarrow \ \beta_3 = \beta_4 = 0, \\ J(e_1, e_1, f_1, e_2) = 0 \ \Rightarrow \ \beta_2 = 0, \\ J(e_1, e_1, f_1, f_2) = 0 \ \Rightarrow \ \xi_1 = \xi_2 = 0, \\ J(e_1, e_2, e_2, f_3) = 0 \ \Rightarrow \ \beta_1(2\beta_1 - 1) = 0. \end{gathered}$$

Hence, we have two Jordan superalgebras with $\beta_1 \in \{0, \frac{1}{2}\}$, which are isomorphic to $\mathcal{J}_5$ and $\mathcal{J}_{11}$.

6. $(\mu_1, \mu_2) = (1, 1)$. In this case, we can consider the action of e_2. By using a simple change of basis, the matrix of L_{e_2} has one of the following forms:

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}.$$

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$ we have:

$$J(e_1, e_1, e_2, f_1) = 0 \Rightarrow \tau_1 = 0, \quad J(e_1, e_1, e_2, f_2) = 0 \Rightarrow \tau_2 = 0,$$
$$J(e_1, e_1, f_1, f_2) = 0 \Rightarrow \xi_2 = 0.$$

Hence, we have the following Jordan superalgebra: $e_1^2 = e_1,\ e_2^2 = e_2,\ e_1f_1 = f_1,\ e_1f_2 = f_2,\ f_1f_2 = \xi_1e_1$ which repeats $\mathcal{J}_2$ and $\mathcal{J}_6$.

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}$ we have:

$$J(e_1, e_1, e_2, f_1) = 0 \Rightarrow f_2 = 0.$$

Thus, there are no superalgebras in this case.

II) Let $L_{e_1} \simeq \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{pmatrix}$ then the rule of multiplication can be written as follows:

$$e_1^2 = e_1, \ e_2^2 = e_2, \ e_1f_1 = \mu_1 f_1 + f_2, \ e_1f_2 = \mu_1 f_2,$$
$$e_2f_1 = \beta_1 f_1 + \beta_2 f_2, \ f_1f_2 = \xi_1e_1 + \xi_2e_2, \ e_2f_2 = \beta_3 f_1 + \beta_4 f_2.$$

From $J(e_1, e_1, e_1, f_1) = 0$, $J(e_1, e_1, e_1, f_2) = 0$, we obtain equations below

$$1 - 6\mu_1 + 6\mu_1^2 = 0, \quad (\mu_1 - 1)\mu_1(2\mu_1 - 1) = 0.$$

Evidently, this is a contradiction.

Let $\mathfrak{J}_0 \cong \mathcal{U}_2 \oplus \mathcal{U}_2$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2) + (\mathbb{F}f_1 + \mathbb{F}f_2)$ with multiplication rules

$e_1f_1 = \alpha_1 f_1 + \alpha_2 f_2, \ e_1f_2 = \alpha_3 f_1 + \alpha_4 f_2, \ e_2f_1 = \beta_1 f_1 + \beta_2 f_2, \ f_1f_2 = \xi_1e_1 + \xi_2e_2, \ e_2f_2 = \beta_3 f_1 + \beta_4 f_2.$

For the action of the operator L_{e_1} on $\mathfrak{J}_1$ we can write the following matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}.$$

However, it is easy to prove that, by using a simple change of basis, the matrix of L_{e_1} has one of the following forms:

$$\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \quad \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{pmatrix}.$$

I) Let $L_{e_1} \simeq \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$, then the rule of multiplication can be written as follows:

$$e_1 f_1 = \mu_1 f_1, \;\; e_1 f_2 = \mu_2 f_2, \;\; e_2 f_1 = \beta_1 f_1 + \beta_2 f_2, \;\; f_1 f_2 = \xi_1 e_1 + \xi_2 e_2,$$
$$e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

Using Jordan super identity we get the following results:

$$J(e_1, e_1, e_1, f_1) = 0 \;\Rightarrow\; \mu_1 = 0, \quad J(e_1, e_1, e_1, f_2) = 0 \;\Rightarrow\; \mu_2 = 0.$$

Thus, we can consider the action of e_2. By using a simple change of basis, the matrix of L_{e_2} has one of the following forms:

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}.$$

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$ we have:

$$J(e_2, e_2, e_2, f_1) = 0 \;\Rightarrow\; \tau_1 = 0, \quad J(e_2, e_2, e_2, f_2) = 0 \;\Rightarrow\; \tau_2 = 0.$$

Hence, we have the Jordan superalgebra: $f_1 f_2 = \xi_1 e_1 + \xi_2 e_2$.

- If $(\xi_1, \xi_2) = (0, 0)$ the superalgebra is trivial.
- If $(\xi_1, \xi_2) \neq (0, 0)$ then by changing the basis as follows

$$e_1' = \xi_1 e_1 + \xi_2 e_2, \; e_2' = \begin{cases} e_2, & \text{if } \xi_1 \neq 0, \\ e_1, & \text{if } \xi_1 = 0. \end{cases}$$

  we get the superalgebra $\mathcal{J}_{17}$.

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}$ we have:

$$J(e_2, e_2, e_2, f_1) = 0 \;\Rightarrow\; \tau_1 = 0.$$

Hence, we have the Jordan superalgebra: $e_2 f_1 = f_2, \; f_1 f_2 = \xi_1 e_1 + \xi_2 e_2$.

- If $(\xi_1, \xi_2) = (0, 0)$, we have the superalgebra $\mathcal{J}_{18}$.
- If $\xi_2 = 0, \xi_1 \neq 0$, then by changing $f_1' = \frac{1}{\sqrt{\xi_1}} f_1, \; f_2' = \frac{1}{\sqrt{\xi_1}} f_2$ we get the superalgebra $\mathcal{J}_{19}$.
- If $\xi_2 \neq 0$, then changing $f_1' = \frac{1}{\sqrt{\xi_2}} f_1, \; f_2' = \frac{1}{\sqrt{\xi_2}} f_2, \; \frac{\xi_1}{\xi_2} = t$ we have

$$e_2 f_1' = f_2', \; f_1' f_2' = t e_1 + e_2.$$

  Further, by changing $e_2' = t e_1 + e_2$ we obtain the superalgebra $\mathcal{J}_{20}$.

Let $\mathfrak{J}_0 \cong \mathcal{U}_1 \oplus \mathcal{U}_2$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2) + (\mathbb{F}f_1 + \mathbb{F}f_2)$ with multiplication rules

$$e_1^2 = e_1, \;\; e_1 f_1 = \alpha_1 f_1 + \alpha_2 f_2, \;\; e_1 f_2 = \alpha_3 f_1 + \alpha_4 f_2, \;\; e_2 f_1 = \beta_1 f_1 + \beta_2 f_2,$$
$$f_1 f_2 = \xi_1 e_1 + \xi_2 e_2, \;\; e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

For the action of the operator L_{e_1} on $\mathfrak{J}_1$ we can write the following matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}.$$

However, it is easy to prove that, by using a simple change of basis, the matrix of L_{e_1} has one of the following forms:

$$\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \quad \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{pmatrix}.$$

I) Let $L_{e_1} \simeq \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ then the rule of multiplication can be written as follows:

$$e_1^2 = e_1,\ e_1 f_1 = \mu_1 f_1,\ \ e_1 f_2 = \mu_2 f_2,\ \ e_2 f_1 = \beta_1 f_1 + \beta_2 f_2,$$
$$f_1 f_2 = \xi_1 e_1 + \xi_2 e_2,\ \ e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

From $J(e_1, e_1, e_1, f_1) = 0$, $J(e_1, e_1, e_1, f_2) = 0$, we obtain equations below

$$(\mu_i - 1)\mu_i(2\mu_i - 1) = 0, \quad i = \overline{1, 2}.$$

Up to permutation of f_1 and f_2 we have six possibilities:

$$(\mu_1, \mu_2) \in \{(0, 0), (0, \tfrac{1}{2}), (0, 1), (\tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{1}{2}, 1), (1, 1)\}.$$

1. $(\mu_1, \mu_2) = (0, 0)$. In this case, we can consider the action of e_2. By using a simple change of basis, the matrix of L_{e_2} has one of the following forms:

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}.$$

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$ we have:

$$J(e_1, e_1, f_1, f_2) = 0 \ \Rightarrow\ \xi_1 = 0,\ \ J(e_2, e_2, e_2, f_1) = 0 \ \Rightarrow\ \tau_1 = 0,$$
$$J(e_2, e_2, e_2, f_2) = 0 \ \Rightarrow\ \tau_2 = 0.$$

So we have the superalgebras $\mathcal{J}_{21}$ and $\mathcal{J}_{22}$. When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}$ we have

$$J(e_1, e_1, f_1, f_2) = 0 \ \Rightarrow\ \xi_1 = 0, \quad J(e_2, e_2, e_2, f_2) = 0 \ \Rightarrow\ \tau_1 = 0.$$

Which gives us the superalgebras $\mathcal{J}_{23}$ and $\mathcal{J}_{24}$.

2. $(\mu_1, \mu_2) = (0, \frac{1}{2})$. In this case, we have the following results

$$J(e_1, e_1, e_2, f_1) = 0 \ \Rightarrow\ \beta_2 = 0,\ \ J(e_1, e_1, f_2, e_2) = 0 \ \Rightarrow\ \beta_3 = 0,$$
$$J(e_1, e_1, f_2, f_1) = 0 \ \Rightarrow\ \xi_1 = \xi_2 = 0,$$
$$J(e_1, e_2, e_2, f_2) = 0 \ \Rightarrow\ \beta_4 = 0,\ \ J(e_2, e_2, e_2, f_1) = 0 \ \Rightarrow\ \beta_1 = 0.$$

Hence, we obtain the superalgebra $\mathcal{J}_{25}$.

3. $(\mu_1, \mu_2) = (0, 1)$. In this case, we have the following results

$$J(e_1, e_2, e_1, f_2) = 0 \Rightarrow \beta_3 = \beta_4 = 0,$$
$$J(e_1, f_1, e_1, f_2) = 0 \Rightarrow \xi_1 = \xi_2 = 0,$$
$$J(e_1, e_1, e_2, f_1) = 0 \Rightarrow \beta_2 = 0, \ J(e_2, e_2, e_2, f_1) = 0 \Rightarrow \beta_1 = 0.$$

Hence, we get the superalgebra $\mathcal{J}_{26}$.

4. $(\mu_1, \mu_2) = (\frac{1}{2}, \frac{1}{2})$. In this case, we can consider the action of e_2. By using a simple change of basis, the matrix of L_{e_2} has one of the following forms:

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}.$$

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$ we have:

$$J(e_1, e_2, e_2, f_1) = 0 \Rightarrow \tau_1 = 0, \ J(e_1, e_2, e_2, f_2) = 0 \Rightarrow \tau_2 = 0.$$

So we have the following superalgebra:

$$e_1^2 = e_1, \ e_1 f_1 = \tfrac{1}{2} f_1, \ e_1 f_2 = \tfrac{1}{2} f_2, \ f_1 f_2 = \xi_1 e_1 + \xi_2 e_2.$$

- If $\xi_1 = \xi_2 = 0$ then we have the superalgebra $\mathcal{J}_{27}$.
- If $\xi_1 = 0, \xi_2 \neq 0$ then by the change $f_1' = \frac{1}{\xi_2} f_1$ we get the superalgebra $\mathcal{J}_{28}$.
- If $\xi_1 \neq 0, \ \xi_2 = 0$, then by changing $f_1' = \frac{1}{\xi_1} f_1$ we get the superalgebra $\mathcal{J}_{29}$.
- If $\xi_1 \neq 0, \ \xi_2 \neq 0$, then by changing $f_1' = \frac{1}{\xi_1} f_1$ and $e_2' = \frac{\xi_2}{\xi_1} e_2$ we obtain the superalgebra $\mathcal{J}_{30}$.

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}$ we have

$$J(e_1, e_2, e_2, f_1) = 0 \Rightarrow \tau_1 = 0, \ J(e_1, e_2, f_1, f_1) = 0 \Rightarrow \xi_1 = 0.$$

So we have the superalgebras $\mathcal{J}_{31}$ and $\mathcal{J}_{32}$.

5. $(\mu_1, \mu_2) = (\frac{1}{2}, 1)$. In this case, we have the following results

$$J(e_1, e_1, e_2, f_2) = 0 \Rightarrow \beta_3 = \beta_4 = 0,$$
$$J(e_1, e_1, f_1, e_2) = 0 \Rightarrow \beta_2 = 0,$$
$$J(e_1, e_1, f_1, f_2) = 0 \Rightarrow \xi_1 = \xi_2 = 0, \ J(e_1, e_2, e_2, f_1) = 0 \Rightarrow \beta_1 = 0.$$

Hence, the obtained superalgebra is $\mathcal{J}_{33}$.

6. $(\mu_1, \mu_2) = (1, 1)$. In this case, we can consider the action of e_2. By using a simple change of basis, the matrix of L_{e_2} has one of the following forms:

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}.$$

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$ we have:

$$J(e_1, e_1, e_2, f_1) = 0 \Rightarrow \tau_1 = 0, \ J(e_1, e_1, e_2, f_2) = 0 \Rightarrow \tau_2 = 0,$$
$$J(e_1, e_1, f_1, f_2) = 0 \Rightarrow \xi_2 = 0.$$

So we have superalgebras $\mathcal{J}_{34}$ and $\mathcal{J}_{35}$.

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}$ we have

$$J(e_1, e_1, e_2, f_1) = 0 \Rightarrow 1 = 0.$$

Which is, obviously, a contradiction.

Let $\mathfrak{J}_0 \cong \mathcal{B}_1$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2) + (\mathbb{F}f_1 + \mathbb{F}f_2)$ with multiplication rules

$$e_1^2 = e_1, \ e_1 f_1 = \alpha_1 f_1 + \alpha_2 f_2, \ e_1 e_2 = e_2, \ e_1 f_2 = \alpha_3 f_1 + \alpha_4 f_2,$$
$$e_2 f_1 = \beta_1 f_1 + \beta_2 f_2, \ f_1 f_2 = \xi_1 e_1 + \xi_2 e_2, \ e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

For the action of the operator L_{e_1} on $\mathfrak{J}_1$ we can write the following matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}.$$

However, it is easy to prove that, by using a simple change of basis, the matrix of L_{e_1} has one of the following forms:

$$\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \quad \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{pmatrix}.$$

I) Let $L_{e_1} \simeq \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ then the rule of multiplication can be written as follows:

$$e_1^2 = e_1, \ e_1 f_1 = \mu_1 f_1, \ e_1 e_2 = e_2, \ e_1 f_2 = \mu_2 f_2, \ e_2 f_1 = \beta_1 f_1 + \beta_2 f_2,$$
$$f_1 f_2 = \xi_1 e_1 + \xi_2 e_2, \ e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

From $J(e_1, e_1, e_1, f_1) = 0$, $J(e_1, e_1, e_1, f_2) = 0$, we obtain equations below

$$(\mu_i - 1)\mu_i(2\mu_i - 1) = 0, \quad i = \overline{1, 2}.$$

Up to permutation of f_1 and f_2 we have six possibilities:

$$(\mu_1, \mu_2) \in \{(0, 0), (0, \tfrac{1}{2}), (0, 1), (\tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{1}{2}, 1), (1, 1)\}.$$

1. $(\mu_1, \mu_2) = (0, 0)$. In this case, we have the following results
$$J(e_1, e_1, e_2, f_1) = 0 \Rightarrow \beta_1 = \beta_2 = 0,$$
$$J(e_1, e_1, e_2, f_2) = 0 \Rightarrow \beta_3 = \beta_4 = 0,$$
$$J(e_1, e_1, f_1, f_2) = 0 \Rightarrow \xi_1 = \xi_2 = 0.$$
Hence, the obtained superalgebra is $\mathcal{J}_{36}$.
2. $(\mu_1, \mu_2) = (0, \frac{1}{2})$. In this case, we have the following results

$$J(e_1,e_1,e_2,f_1)=0 \Rightarrow \beta_1=\beta_2=0,$$
$$J(e_1,e_1,f_2,e_2)=0 \Rightarrow \beta_3=0, \ J(e_1,e_2,e_2,f_2)=0 \Rightarrow \beta_4=0,$$
$$J(e_1,e_1,f_2,f_1)=0 \Rightarrow \xi_1=\xi_2=0.$$

Hence, the obtained superalgebra is $\mathcal{J}_{37}$.

3. $(\mu_1,\mu_2)=(0,1)$. In this case, we have the following results

$$J(e_1,e_2,e_1,f_1)=0 \Rightarrow \beta_1=\beta_2=0,$$
$$J(e_1,e_1,e_2,f_2)=0 \Rightarrow \beta_3=0,$$
$$J(e_1,e_1,f_1,f_2)=0 \Rightarrow \xi_1=\xi_2=0,$$
$$J(f_2,e_2,e_2,e_2)=0 \Rightarrow \beta_4=0.$$

Hence, the obtained superalgebra is $\mathcal{J}_{38}$.

4. $(\mu_1,\mu_2)=(\frac{1}{2},\frac{1}{2})$. In this case, we can consider the action of e_2. By using a simple change of basis, the matrix of L_{e_2} have one of the following forms:

$$\begin{pmatrix}\tau_1 & 0\\ 0 & \tau_2\end{pmatrix}, \quad \begin{pmatrix}\tau_1 & 1\\ 0 & \tau_1\end{pmatrix}.$$

When $L_{e_2} \simeq \begin{pmatrix}\tau_1 & 0\\ 0 & \tau_2\end{pmatrix}$ we have:

$$J(e_1,e_2,e_2,f_1)=0 \Rightarrow \tau_1=0, \ J(e_1,e_2,e_2,f_2)=0 \Rightarrow \tau_2=0,$$
$$J(e_1,e_2,f_1,f_2)=0 \Rightarrow \xi_1=0.$$

So we have the superalgebras $\mathcal{J}_{39}$ and $\mathcal{J}_{40}$.

When $L_{e_2} \simeq \begin{pmatrix}\tau_1 & 1\\ 0 & \tau_1\end{pmatrix}$ we have

$$J(e_1,e_2,e_2,f_1)=0 \Rightarrow \tau_1=0, \ J(e_1,e_2,f_1,f_2)=0 \Rightarrow \xi_1=0.$$

which gives us the superalgebras $\mathcal{J}_{41}$ and $\mathcal{J}_{42}$.

5. $(\mu_1,\mu_2)=(\frac{1}{2},1)$. In this case, we have the following results

$$J(e_1,e_1,e_2,f_2)=0 \Rightarrow \beta_3=0, \ J(e_1,e_1,f_1,e_2)=0 \Rightarrow \beta_2=0,$$
$$J(e_1,e_1,f_1,f_2)=0 \Rightarrow \xi_1=\xi_2=0, J(e_1,e_2,e_2,f_1)=0 \Rightarrow \beta_1=0,$$
$$J(e_2,e_2,e_2,f_2)=0 \Rightarrow \beta_4=0.$$

Hence, the obtained superalgebra is $\mathcal{J}_{43}$.

6. $(\mu_1,\mu_2)=(1,1)$. In this case, we can consider the action of e_2. By using a simple change of basis, the matrix of L_{e_2} have one of the following forms:

$$\begin{pmatrix}\tau_1 & 0\\ 0 & \tau_2\end{pmatrix}, \quad \begin{pmatrix}\tau_1 & 1\\ 0 & \tau_1\end{pmatrix}.$$

When $L_{e_2} \simeq \begin{pmatrix}\tau_1 & 0\\ 0 & \tau_2\end{pmatrix}$ we have:

$$J(e_2,e_2,e_2,f_1)=0 \Rightarrow \tau_1=0, \ J(e_2,e_2,e_2,f_2)=0 \Rightarrow \tau_2=0.$$

So we have the following superalgebra:

$$e_1^2=e_1, \ e_1f_1=f_1, \ e_1e_2=e_2, \ e_1f_2=f_2, \ f_1f_2=\xi_1e_1+\xi_2e_2,$$

- If $(\xi_1, \xi_2) = (0, 0)$ we have the superalgebra $\mathcal{J}_{44}$.
- If $(\xi_1, \xi_2) \neq (0, 0)$
  - (a) When $\xi_1 = 0$, by changing $f_1' = \frac{1}{\xi_2} f_1$ we get the superalgebra $\mathcal{J}_{45}$.
  - (b) When $\xi_1 \neq 0$, by changing $f_1' = \frac{1}{\xi_1} f_1$ and denoting $\frac{\xi_2}{\xi_1} = t$ we can write

$$e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 f_1 = f_1,\ e_1 f_2 = f_2,\ f_1 f_2 = e_1 + t e_2.$$

In this case, we get $\mathcal{J}_{46}$ when $t = 0$ and $\mathcal{J}_{47}$ when $t \neq 0$.

When $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}$ we have

$$J(e_2, e_2, e_2, f_2) = 0 \ \Rightarrow\ \tau_1 = 0,\ \ J(e_2, e_2, f_1, f_1) = 0 \ \Rightarrow\ \xi_1 = 0.$$

So we have the superalgebras $\mathcal{J}_{48}$ and $\mathcal{J}_{49}$.

II) Let $L_{e_1} \simeq \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{pmatrix}$ then the rule of multiplication can be written as follows:

$$e_1^2 = e_1,\ e_1 e_2 = e_2,\ e_1 f_1 = \mu_1 f_1 + f_2,\ e_1 f_2 = \mu_1 f_2,$$

$$e_2 f_1 = \beta_1 f_1 + \beta_2 f_2,\ f_1 f_2 = \xi_1 e_1 + \xi_2 e_2,\ e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

However, from

$$\begin{gathered} J(e_1, e_1, e_1, f_1) = 0 \ \Rightarrow\ 1 - 6\mu_1 + 6\mu_1^2 = 0, \\ J(e_1, e_1, e_1, f_2) = 0 \ \Rightarrow\ (\mu_1 - 1)\mu_1(2\mu_1 - 1) = 0. \end{gathered}$$

we get a contradiction.

Let $\mathfrak{J}_0 \cong \mathcal{B}_2$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2) + (\mathbb{F}f_1 + \mathbb{F}f_2)$ with multiplication rules

$$e_1^2 = e_1,\ e_1 f_1 = \alpha_1 f_1 + \alpha_2 f_2,\ e_1 e_2 = \frac{1}{2} e_2,\ e_1 f_2 = \alpha_3 f_1 + \alpha_4 f_2,$$

$$e_2 f_1 = \beta_1 f_1 + \beta_2 f_2,\ f_1 f_2 = \xi_1 e_1 + \xi_2 e_2,\ e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

For the action of the operator L_{e_1} on $\mathfrak{J}_1$ we can write the following matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}.$$

However, it is easy to prove that, by using a simple change of basis, the matrix of L_{e_1} has one of the following forms:

$$\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix},\quad \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{pmatrix}.$$

I) Let $L_{e_1} \simeq \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ then the rule of multiplication can be written as follows:

$$e_1^2 = e_1, \; e_1 f_1 = \mu_1 f_1, \; e_1 e_2 = \frac{1}{2} e_2, \; e_1 f_2 = \mu_2 f_2,$$
$$e_2 f_1 = \beta_1 f_1 + \beta_2 f_2, \; f_1 f_2 = \xi_1 e_1 + \xi_2 e_2, \; e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

From $J(e_1, e_1, e_1, f_1) = 0$, $J(e_1, e_1, e_1, f_2) = 0$, we obtain equations below

$$(\mu_i - 1)\mu_i(2\mu_i - 1) = 0, \quad i = \overline{1, 2}.$$

Up to permutation of f_1 and f_2 we have six possibilities:

$$(\mu_1, \mu_2) \in \{(0, 0), (0, \tfrac{1}{2}), (0, 1), (\tfrac{1}{2}, \tfrac{1}{2}), (\tfrac{1}{2}, 1), (1, 1)\}.$$

1. $(\mu_1, \mu_2) = (0, 0)$. In this case, we have the following results
$$J(e_1, e_1, e_2, f_1) = 0 \;\Rightarrow\; \beta_1 = \beta_2 = 0,$$
$$J(e_1, e_1, e_2, f_2) = 0 \;\Rightarrow\; \beta_3 = \beta_4 = 0,$$
$$J(e_1, e_1, f_1, f_2) = 0 \;\Rightarrow\; \xi_1 = \xi_2 = 0.$$
Hence, the obtained superalgebra is $\mathcal{J}_{50}$.
2. $(\mu_1, \mu_2) = (0, \frac{1}{2})$. In this case, we have the following results
$$J(e_1, e_1, e_2, f_1) = 0 \;\Rightarrow\; \beta_1 = 0, \;\; J(e_1, e_1, f_2, f_1) = 0 \;\Rightarrow\; \xi_1 = 0,$$
$$J(e_1, e_2, e_1, f_2) = 0 \;\Rightarrow\; \beta_4 = 0, \;\; J(e_1, e_2, e_2, f_1) = 0 \;\Rightarrow\; \beta_2\beta_3 = 0.$$
We consider the following subcases:
   (a) $\beta_2 = 0$. If $\beta_3 = 0$ then we take the superalgebra $\mathcal{J}_{51}$ when $\xi_2 = 0$, and when $\xi_2 \neq 0$ we can change $f_1' = \frac{1}{\xi_2} f_1$ and thereby obtain superalgebra $\mathcal{J}_{52}$.
   If $\beta_3 \neq 0$, we get the superalgebra $\mathcal{J}_{53}$ when $\xi_2 = 0$, and when $\xi_2 \neq 0$ then we change the basis by taking $f_1' = \sqrt{\frac{\beta_3}{\xi_2}} f_1, \; f_2' = \frac{1}{\sqrt{\beta_3 \xi_2}} f_2$, which gives us the superalgebra $\mathcal{J}_{54}$.
   (b) $\beta_3 = 0, \; \beta_2 \neq 0$. In this subcase we obtain superalgebras $\mathcal{J}_{55}$ (when $\xi_2 = 0$) and $\mathcal{J}_{56}$ (when $\xi_2 \neq 0$).
3. $(\mu_1, \mu_2) = (0, 1)$. In this case, we have the following results
$$J(e_1, e_1, e_2, f_1) = 0 \;\Rightarrow\; \beta_1 = \beta_2 = 0,$$
$$J(e_1, e_1, e_2, f_2) = 0 \;\Rightarrow\; \beta_3 = \beta_4 = 0,$$
$$J(e_1, e_1, f_1, f_2) = 0 \;\Rightarrow\; \xi_1 = \xi_2 = 0.$$
This gives us the Jordan superalgebra $\mathcal{J}_{57}$.
4. $(\mu_1, \mu_2) = (\frac{1}{2}, \frac{1}{2})$. In this case, we have the following results
$$J(e_1, e_2, e_1, f_1) = 0 \;\Rightarrow\; \beta_1 = \beta_2 = 0,$$
$$J(e_1, e_2, e_1, f_2) = 0 \;\Rightarrow\; \beta_3 = \beta_4 = 0,$$
$$J(e_1, f_1, e_1, f_2) = 0 \;\Rightarrow\; \xi_2 = 0, \;\; J(e_2, f_1, e_1, f_2) = 0 \;\Rightarrow\; \xi_1 = 0.$$
This gives us the Jordan superalgebra $\mathcal{J}_{58}$.
5. $(\mu_1, \mu_2) = (\frac{1}{2}, 1)$. In this case, we have the following results
$$J(e_1, e_1, e_2, f_2) = 0 \;\Rightarrow\; \beta_4 = 0, \;\; J(e_1, e_1, f_1, f_2) = 0 \;\Rightarrow\; \xi_1 = 0,$$
$$J(e_1, e_2, e_1, f_1) = 0 \;\Rightarrow\; \beta_1 = 0, \;\; J(e_1, e_2, e_2, f_2) = 0 \;\Rightarrow\; \beta_2\beta_3 = 0.$$

This case, being similar to the second one, gives us the Jordan superalgebras $\mathcal{J}_{59} - \mathcal{J}_{64}$.

6. $(\mu_1, \mu_2) = (1, 1)$. In this case, we have the following results

$$J(e_1, e_1, e_2, f_1) = 0 \;\Rightarrow\; \beta_1 = \beta_2 = 0,$$
$$J(e_1, e_1, e_2, f_2) = 0 \;\Rightarrow\; \beta_3 = \beta_4 = 0,$$
$$J(e_1, e_1, f_1, f_2) = 0 \;\Rightarrow\; \xi_2 = 0, \; J(e_2, f_1, e_1, f_2) = 0 \;\Rightarrow\; \xi_1 = 0.$$

Here we obtain the Jordan superalgebra $\mathcal{J}_{65}$.

II) Let $L_{e_1} \simeq \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{pmatrix}$ then the rule of multiplication can be written as follows:

$$e_1^2 = e_1, \; e_1 f_1 = \mu_1 f_1 + f_2, \; e_1 e_2 = \frac{1}{2} e_2, \; e_1 f_2 = \mu_1 f_2, \; e_2^2 = 0,$$
$$e_2 f_1 = \beta_1 f_1 + \beta_2 f_2, \; f_1 f_2 = \xi_1 e_1 + \xi_2 e_2, \; e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

However, from the following

$$J(e_1, e_1, e_1, f_1) = 0 \;\Rightarrow\; 1 - 6\mu_1 + 6\mu_1^2 = 0,$$
$$J(e_1, e_1, e_1, f_2) = 0 \;\Rightarrow\; (\mu_1 - 1)\mu_1(2\mu_1 - 1) = 0,$$

we get a contradiction.

Let $\mathfrak{J}_0 \cong \mathcal{B}_3$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2) + (\mathbb{F}f_1 + \mathbb{F}f_2)$ with multiplication rules

$$e_1^2 = e_2, \; e_1 f_1 = \alpha_1 f_1 + \alpha_2 f_2, \; e_1 f_2 = \alpha_3 f_1 + \alpha_4 f_2,$$
$$e_2 f_1 = \beta_1 f_1 + \beta_2 f_2, \; f_1 f_2 = \xi_1 e_1 + \xi_2 e_2, \; e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

For the action of the operator L_{e_1} on $\mathfrak{J}_1$ we can write the following matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}.$$

However, it is easy to prove that, by using a simple change of basis, the matrix of L_{e_1} has one of the following forms:

$$\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \quad \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{pmatrix}.$$

I) Let $L_{e_1} \simeq \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$ then the rule of multiplication can be written as follows:

$$e_1^2 = e_2, \;\; e_1 f_1 = \mu_1 f_1, \;\; e_1 f_2 = \mu_2 f_2, \;\; e_2 f_1 = \beta_1 f_1 + \beta_2 f_2,$$
$$f_1 f_2 = \xi_1 e_1 + \xi_2 e_2, \;\; e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

Let's assume that $\mu_1 \neq 0$. Then, from

$$J(e_1, e_1, e_1, f_1) = 0 \;\Rightarrow\; \beta_2 = 0, \;\; J(e_2, e_2, e_2, f_1) = 0 \;\Rightarrow\; \beta_1 = 0,$$
$$J(e_1, e_1, e_1, f_1) = 0 \;\Rightarrow\; \mu_1 = 0,$$

results, we get a contradiction. So $\mu_1 = 0$.

Further, let's assume that $\mu_2 \neq 0$. Then, from

$$J(e_1, e_1, e_1, f_2) = 0 \Rightarrow \beta_3 = 0, \ J(e_1, e_1, e_2, f_1) = 0 \Rightarrow \beta_1 = 0,$$
$$J(e_2, e_2, e_2, f_2) = 0 \Rightarrow \beta_4 = 0, \ J(e_1, e_1, e_1, f_2) = 0 \Rightarrow \mu_2 = 0.$$

results we get a contradiction again. So $\mu_2 = 0$.

With obtained results we can consider the action of the operator L_{e_2} on $\mathfrak{J}_1$ which can be written in one of the following forms:

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}.$$

- $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$ gives the following results:
  $$J(e_1, e_1, e_2, f_1) = 0 \Rightarrow \tau_1 = 0, \ J(e_1, e_1, e_2, f_2) = 0 \Rightarrow \tau_2 = 0,$$
  by which we obtain the following superalgebra:
  $$e_1^2 = e_2, \quad f_1 f_2 = \xi_1 e_1 + \xi_2 e_2.$$
  a) If $\xi_1 = 0, \xi_2 = 0$ we have the superalgebra $\mathcal{J}_{66}$.
  b) If $\xi_1 = 0, \xi_2 \neq 0$, then by changing $f_1' = \frac{1}{\xi_2} f_1$ we get the superalgebra $\mathcal{J}_{67}$.
  c) If $\xi_1 \neq 0$, then by changing $f_1' = \frac{1}{\xi_1} f_1$ and $e_1' = e_1 + \frac{\xi_2}{\xi_1} e_2$ we obtain the superalgebra $\mathcal{J}_{68}$.
- $L_{e_2} \simeq \begin{pmatrix} \tau_1 & 1 \\ 0 & \tau_1 \end{pmatrix}$ gives the following results:
  $$J(e_1, e_1, e_2, f_1) = 0 \Rightarrow \tau_1 = 0, \ J(e_1, e_1, f_1, f_1) = 0 \Rightarrow \xi_1 = \xi_2 = 0,$$
  by which we obtain $\mathcal{J}_{69}$.

II) Let $L_{e_1} \simeq \begin{pmatrix} \mu_1 & 1 \\ 0 & \mu_1 \end{pmatrix}$ then the rule of multiplication can be written as follows:

$$e_1^2 = e_2, \ e_1 f_1 = \mu_1 f_1 + f_2, \ e_1 f_2 = \mu_1 f_2, \ e_2 f_1 = \beta_1 f_1 + \beta_2 f_2,$$
$$f_1 f_2 = \xi_1 e_1 + \xi_2 e_2, \ e_2 f_2 = \beta_3 f_1 + \beta_4 f_2.$$

Then we have the following results:

$$J(e_1, e_1, f_1, e_1) = 0 \Rightarrow \beta_3 = 0, \ J(e_2, e_2, e_2, f_2) = 0 \Rightarrow \beta_4 = 0,$$
$$J(e_1, e_1, e_1, f_2) = 0 \Rightarrow \mu_1 = 0, \ J(e_1, e_1, f_1, e_1) = 0 \Rightarrow \beta_1 = 0,$$
$$J(e_1, e_1, f_1, f_1) = 0 \Rightarrow \beta_2 \xi_1 = 0, \ \beta_2 \xi_2 = 2\xi_1.$$

If $\beta_2 = 0$ then we have $\xi_1 = 0$ and get the superalgebras $\mathcal{J}_{70}$ and $\mathcal{J}_{71}$.
If $\beta_2 \neq 0$ then we have $\xi_1 = \xi_2 = 0$. Then by changing the basis as $e_1' = e_1 - \frac{1}{\beta_2} e_2, \ f_2' = \beta_2 f_2$ we obtain the superalgebra

$$e_1'^2 = e_2, \ e_2 f_1 = f_2'.$$

However, this superalgebra is isomorphic to $\mathcal{J}_{69}$. □

Theorem 11. *Up to isomorphism there are 59 Jordan superalgebras of type* $(3,1)$*, which are presented below with some additional information:*

№	*Multiplication rules*	*Decomposition*
$\mathfrak{J}_1$	$e_1^2=e_1,\ e_2^2=e_2,\ e_3^2=e_3$	$\mathcal{U}_1\oplus\mathcal{U}_1\oplus\mathcal{U}_1\oplus\mathcal{S}_1^1$
$\mathfrak{J}_2$	$e_1^2=e_1,\ e_2^2=e_2,\ e_3^2=e_3,\ e_1f=f$	$\mathcal{S}_2^2\oplus\mathcal{U}_1\oplus\mathcal{U}_1$
$\mathfrak{J}_3$	$e_1^2=e_1,\ e_2^2=e_2,\ e_3^2=e_3,\ e_1f=\frac{1}{2}f$	$\mathcal{S}_1^2\oplus\mathcal{U}_1\oplus\mathcal{U}_1$
$\mathfrak{J}_4$	$e_1^2=e_1,\ e_2^2=e_2,\ e_3^2=e_3,\ e_1f=\frac{1}{2}f,\ e_2f=\frac{1}{2}f$	$\mathcal{S}_{13}^3\oplus\mathcal{U}_1$
$\mathfrak{J}_5$	$e_1^2=e_1,\ e_2^2=e_2$	$\mathcal{U}_1\oplus\mathcal{U}_1\oplus\mathcal{U}_2\oplus\mathcal{S}_1^1$
$\mathfrak{J}_6$	$e_1^2=e_1,\ e_2^2=e_2,\ e_1f=f$	$\mathcal{S}_2^2\oplus\mathcal{U}_1\oplus\mathcal{S}_1^1$
$\mathfrak{J}_7$	$e_1^2=e_1,\ e_2^2=e_2,\ e_1f=\frac{1}{2}f$	$\mathcal{S}_1^2\oplus\mathcal{U}_1\oplus\mathcal{S}_1^1$
$\mathfrak{J}_8$	$e_1^2=e_1,\ e_2^2=e_2,\ e_1f=\frac{1}{2}f,\ e_2f=\frac{1}{2}f$	$\mathcal{S}_{13}^3\oplus\mathcal{S}_1^1$
$\mathfrak{J}_9$	$e_1^2=e_1$	$\mathcal{U}_1\oplus\mathcal{U}_2\oplus\mathcal{U}_2\oplus\mathcal{S}_1^1$
$\mathfrak{J}_{10}$	$e_1^2=e_1,\ e_1f=f$	$\mathcal{S}_2^2\oplus\mathcal{U}_2\oplus\mathcal{S}_1^1$
$\mathfrak{J}_{11}$	$e_1^2=e_1,\ e_1f=\frac{1}{2}f$	$\mathcal{S}_1^2\oplus\mathcal{U}_2\oplus\mathcal{S}_1^1$
$\mathfrak{J}_{12}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_3^2=e_3$	$\mathcal{B}_1\oplus\mathcal{U}_1\oplus\mathcal{S}_1^1$
$\mathfrak{J}_{13}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_3^2=e_3,\ e_3f=f$	$\mathcal{B}_1\oplus\mathcal{S}_2^2$
$\mathfrak{J}_{14}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_3^2=e_3,\ e_3f=\frac{1}{2}f$	$\mathcal{B}_1\oplus\mathcal{S}_2^1$
$\mathfrak{J}_{15}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_3^2=e_3,\ e_1f=f$	$\mathcal{S}_{10}^3\oplus\mathcal{U}_1$
$\mathfrak{J}_{16}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_3^2=e_3,\ e_1f=\frac{1}{2}f$	$\mathcal{S}_9^3\oplus\mathcal{U}_1$
$\mathfrak{J}_{17}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_3^2=e_3,\ e_1f=\frac{1}{2}f,\ e_3f=\frac{1}{2}f$	*Indecomposable*
$\mathfrak{J}_{18}$	$e_1^2=e_1,\ e_1e_2=e_2$	$\mathcal{B}_1\oplus\mathcal{U}_2\oplus\mathcal{S}_1^1$
$\mathfrak{J}_{19}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_1f=f$	$\mathcal{S}_{10}^3\oplus\mathcal{U}_2^1$
$\mathfrak{J}_{20}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_1f=\frac{1}{2}f$	$\mathcal{S}_9^3\oplus\mathcal{U}_2^1$
$\mathfrak{J}_{21}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_3^2=e_3$	$\mathcal{B}_2\oplus\mathcal{U}_1\oplus\mathcal{S}_1^1$
$\mathfrak{J}_{22}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_3^2=e_3,\ e_3f=f$	$\mathcal{B}_2\oplus\mathcal{S}_2^2$
$\mathfrak{J}_{23}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_3^2=e_3,\ e_3f=\frac{1}{2}f$	$\mathcal{B}_2\oplus\mathcal{S}_1^1$
$\mathfrak{J}_{24}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_3^2=e_3,\ e_1f=f$	$\mathcal{S}_{12}^3\oplus\mathcal{U}_1$
$\mathfrak{J}_{25}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_3^2=e_3,\ e_1f=\frac{1}{2}f$	$\mathcal{S}_{11}^3\oplus\mathcal{U}_1$
$\mathfrak{J}_{26}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_3^2=e_3,\ e_1f=\frac{1}{2}f,\ e_3f=\frac{1}{2}f$	*Indecomposable*
$\mathfrak{J}_{27}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2$	$\mathcal{B}_2\oplus\mathcal{U}_2\oplus\mathcal{S}_1^1$
$\mathfrak{J}_{28}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_1f=f$	$\mathcal{S}_{12}^3\oplus\mathcal{U}_2$
$\mathfrak{J}_{29}$	$e_1^2=e_1,\ e_1e_2=\frac{1}{2}e_2,\ e_1f=\frac{1}{2}f$	$\mathcal{S}_{11}^3\oplus\mathcal{U}_2^1$
$\mathfrak{J}_{30}$	$e_1^2=e_2,\ e_3^2=e_3$	$\mathcal{B}_3\oplus\mathcal{U}_1\oplus\mathcal{S}_1^1$
$\mathfrak{J}_{31}$	$e_1^2=e_2,\ e_3^2=e_3,\ e_3f=f$	$\mathcal{S}_2^2\oplus\mathcal{U}_1\oplus\mathcal{U}_2$
$\mathfrak{J}_{32}$	$e_1^2=e_2,\ e_3^2=e_3,\ e_3f=\frac{1}{2}f$	$\mathcal{S}_1^2\oplus\mathcal{U}_1\oplus\mathcal{U}_2$
$\mathfrak{J}_{33}$	$e_1^2=e_2$	$\mathcal{B}_3\oplus\mathcal{U}_2\oplus\mathcal{S}_1^1$
$\mathfrak{J}_{34}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_1e_3=e_3,\ e_2^2=e_3$	$\mathcal{T}_1\oplus\mathcal{S}_1^1$
$\mathfrak{J}_{35}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_1e_3=e_3,\ e_2^2=e_3,\ e_1f=f$	*Indecomposable*
$\mathfrak{J}_{36}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_1e_3=e_3,\ e_2^2=e_3,\ e_1f=\frac{1}{2}f$	*Indecomposable*
$\mathfrak{J}_{37}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_1e_3=e_3$	$\mathcal{T}_2\oplus\mathcal{S}_1^1$
$\mathfrak{J}_{38}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_1e_3=e_3,\ e_1f=f$	*Indecomposable*
$\mathfrak{J}_{39}$	$e_1^2=e_1,\ e_1e_2=e_2,\ e_1e_3=e_3,\ e_1f=\frac{1}{2}f$	*Indecomposable*
$\mathfrak{J}_{40}$	$e_1^2=e_2,\ e_1e_2=e_3$	$\mathcal{T}_3\oplus\mathcal{S}_1^1$
$\mathfrak{J}_{41}$	$e_1^2=e_2,\ e_1e_3=e_2$	$\mathcal{T}_4\oplus\mathcal{S}_1^1$
$\mathfrak{J}_{42}$	$e_1^2=e_1,\ e_2^2=e_2,\ e_3^2=e_1+e_2,\ e_1e_3=\frac{1}{2}e_3,\ e_2e_3=\frac{1}{2}e_3$	$\mathcal{T}_5\oplus\mathcal{S}_1^1$

$\mathfrak{J}_{43}$	$e_1^2 = e_1,\ e_2^2 = e_2,\ e_3^2 = e_1 + e_2,\ e_1e_3 = \frac{1}{2}e_3,\ e_2e_3 = \frac{1}{2}e_3,$ $e_1f = \frac{1}{2}f,\ e_2f = \frac{1}{2}f$	*Indecomposable*
$\mathfrak{J}_{44}$	$e_1^2 = e_1,\ e_1e_2 = \frac{1}{2}e_2,\ e_1e_3 = e_3$	$\mathcal{T}_6 \oplus \mathcal{S}_1^1$
$\mathfrak{J}_{45}$	$e_1^2 = e_1,\ e_1e_2 = \frac{1}{2}e_2,\ e_1e_3 = e_3,\ e_1f = f$	*Indecomposable*
$\mathfrak{J}_{46}$	$e_1^2 = e_1,\ e_1e_2 = \frac{1}{2}e_2,\ e_1e_3 = e_3,\ e_1f = \frac{1}{2}f$	*Indecomposable*
$\mathfrak{J}_{47}$	$e_1^2 = e_1,\ e_1e_2 = \frac{1}{2}e_2,\ e_1e_3 = \frac{1}{2}e_3$	$\mathcal{T}_7 \oplus \mathcal{S}_1^1$
$\mathfrak{J}_{48}$	$e_1^2 = e_1,\ e_1e_2 = \frac{1}{2}e_2,\ e_1e_3 = \frac{1}{2}e_3,\ e_1f = f$	*Indecomposable*
$\mathfrak{J}_{49}$	$e_1^2 = e_1,\ e_1e_2 = \frac{1}{2}e_2,\ e_1e_3 = \frac{1}{2}e_3,\ e_1f = \frac{1}{2}f$	*Indecomposable*
$\mathfrak{J}_{50}$	$e_1^2 = e_1,\ e_1e_2 = \frac{1}{2}e_2,\ e_2^2 = e_3$	$\mathcal{T}_8 \oplus \mathcal{S}_1^1$
$\mathfrak{J}_{51}$	$e_1^2 = e_1,\ e_1e_2 = \frac{1}{2}e_2,\ e_2^2 = e_3,\ e_1f = f$	*Indecomposable*
$\mathfrak{J}_{52}$	$e_1^2 = e_1,\ e_1e_2 = \frac{1}{2}e_2,\ e_2^2 = e_3,\ e_1f = \frac{1}{2}f$	*Indecomposable*
$\mathfrak{J}_{53}$	$e_1^2 = e_1,\ e_1e_2 = \frac{1}{2}e_2,\ e_2^2 = e_3,\ e_1e_3 = e_3$	$\mathcal{T}_9 \oplus \mathcal{S}_1^1$
$\mathfrak{J}_{54}$	$e_1^2 = e_1,\ e_1e_2 = \frac{1}{2}e_2,\ e_2^2 = e_3,\ e_1e_3 = e_3,\ e_1f = f$	*Indecomposable*
$\mathfrak{J}_{55}$	$e_1^2 = e_1,\ e_1e_2 = \frac{1}{2}e_2,\ e_2^2 = e_3,\ e_1e_3 = e_3,\ e_1f = \frac{1}{2}f$	*Indecomposable*
$\mathfrak{J}_{56}$	$e_1^2 = e_1,\ e_2^2 = e_2,\ e_1e_3 = \frac{1}{2}e_3,\ e_2e_3 = \frac{1}{2}e_3$	$\mathcal{T}_{10} \oplus \mathcal{S}_1^1$
$\mathfrak{J}_{57}$	$e_1^2 = e_1,\ e_2^2 = e_2,\ e_1e_3 = \frac{1}{2}e_3,\ e_2e_3 = \frac{1}{2}e_3,\ e_1f = f$	*Indecomposable*
$\mathfrak{J}_{58}$	$e_1^2 = e_1,\ e_2^2 = e_2,\ e_1e_3 = \frac{1}{2}e_3,\ e_2e_3 = \frac{1}{2}e_3,\ e_1f = \frac{1}{2}f$	*Indecomposable*
$\mathfrak{J}_{59}$	$e_1^2 = e_1,\ e_2^2 = e_2,\ e_1e_3 = \frac{1}{2}e_3,\ e_2e_3 = \frac{1}{2}e_3,\ e_1f = \frac{1}{2}f,\ e_2f = \frac{1}{2}f$	*Indecomposable*

Proof. Let $\mathfrak{J}_0 \cong \mathcal{U}_1 \oplus \mathcal{U}_1 \oplus \mathcal{U}_1$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1,\ \ e_3^2 = e_3,\ \ e_2^2 = e_2,\ \ e_if_1 = \beta_if_1,\ i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_1,e_1,e_1,f_1) = 0 \ \Rightarrow\ (\beta_1 - 1)\beta_1(2\beta_1 - 1) = 0,$$
$$J(e_2,e_2,e_2,f_1) = 0 \ \Rightarrow\ (\beta_2 - 1)\beta_2(2\beta_2 - 1) = 0,$$
$$J(e_3,e_3,e_3,f_1) = 0 \ \Rightarrow\ (\beta_3 - 1)\beta_3(2\beta_3 - 1) = 0,$$
$$J(e_1,e_2,e_3,f_1) = 0 \ \Rightarrow\ \beta_1\beta_2\beta_3 = 0.$$

- If $\beta_1 = 1$ then from $J(e_1,e_1,e_2,f_1) = 0$ and $J(e_1,e_1,e_3,f_1) = 0$ we get $\beta_2 = 0$ and $\beta_3 = 0$, respectively, which gives us the superalgebra $\mathfrak{J}_2$.
- If $\beta_1 = 0$ then
  - a) When $\beta_2 = 0$, from $J(e_3,e_3,e_3,f_1) = 0$ we get three superalgebras with $\beta_3 \in \{0,1,\frac{1}{2}\}$. While $\beta_3 = 1$ gives a superalgebra that is isomorphic to $\mathfrak{J}_2$, from $\beta_3 \in \{0,\frac{1}{2}\}$ we obtain superalgebras $\mathfrak{J}_1$ and $\mathfrak{J}_3$.
  - b) When $\beta_2 = 1$, from $J(e_2,e_2,e_3,f_1) = 0$ we get $\beta_3 = 0$, which gives $\mathfrak{J}_2$.
  - c) When $\beta_2 = \frac{1}{2}$, from $J(e_2,e_3,e_3,f_1) = 0$ we get $\beta_3 \in \{0,\frac{1}{2}\}$, which gives superalgebras isomorphic to $\mathfrak{J}_3$ and $\mathfrak{J}_4$.
- If $\beta_1 = \frac{1}{2}$ from $J(e_1,e_2,e_2,f_1) = 0$ we get either $\beta_2 = 0$ or $\beta_2 = \frac{1}{2}$. When the former occurs, we have two superalgebras with $\beta_3 \in \{0,\frac{1}{2}\}$, and when the latter does we have $\beta_3 = 0$. However, all superalgebras obtained here are isomorphic to those of previous steps.

Let $\mathfrak{J}_0 \cong \mathcal{U}_1 \oplus \mathcal{U}_1 \oplus \mathcal{U}_2$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1 \quad e_2^2 = e_2 \quad e_i f_1 = \beta_i f_1,\ i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_1, e_1, e_1, f_1) = 0 \ \Rightarrow\ (\beta_1 - 1)\beta_1(2\beta_1 - 1) = 0,$$

$$J(e_2, e_2, e_2, f_1) = 0 \ \Rightarrow\ (\beta_2 - 1)\beta_2(2\beta_2 - 1) = 0,$$
$$J(e_3, e_3, e_3, f_1) = 0 \ \Rightarrow\ \beta_3 = 0.$$

- If $\beta_1 = 1$ then from $J(e_1, e_1, e_2, f_1) = 0$ we get $\beta_2 = 0$, which gives $\mathfrak{J}_6$.
- If $\beta_1 = 0$ then from $J(e_2, e_2, e_2, f_1) = 0$ we get three superalgebras with $\beta_2 \in \{0, 1, \frac{1}{2}\}$. New superalgebras here are $\mathfrak{J}_5$ and $\mathfrak{J}_7$.
- If $\beta_1 = \frac{1}{2}$ from $J(e_1, e_2, e_2, f_1) = 0$ we get either $\beta_2 = 0$ or $\beta_2 = \frac{1}{2}$. The only new superlgebra here is $\mathfrak{J}_8$.

Let $\mathfrak{J}_0 \cong \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \mathcal{U}_2$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1, \quad e_i f_1 = \beta_i f_1,\ i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_1, e_1, e_1, f_1) = 0 \ \Rightarrow\ (\beta_1 - 1)\beta_1(2\beta_1 - 1) = 0,$$

$$J(e_2, e_2, e_2, f_1) = 0 \ \Rightarrow\ \beta_2 = 0,\ \ J(e_3, e_3, e_3, f_1) = 0 \ \Rightarrow\ \beta_3 = 0.$$

In this case we have three superalgebras with $\beta_1 \in \{0, 1, \frac{1}{2}\}$, which give us $\mathfrak{J}_9, \mathfrak{J}_{10}$ and $\mathfrak{J}_{11}$.

Let $\mathfrak{J}_0 \cong \mathcal{U}_2 \oplus \mathcal{U}_2 \oplus \mathcal{U}_2$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_i f_1 = \beta_i f_1,\ i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_1, e_1, e_1, f_1) = 0 \ \Rightarrow\ \beta_1 = 0,\ \ J(e_2, e_2, e_2, f_1) = 0 \ \Rightarrow\ \beta_2 = 0,$$
$$J(e_3, e_3, e_3, f_1) = 0 \ \Rightarrow\ \beta_3 = 0.$$

So in this case we obtain a trivial superalgebra.

Let $\mathfrak{J}_0 \cong \mathcal{B}_1 \oplus \mathcal{U}_1$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1,\ \ e_3^2 = e_3,\ \ e_1 e_2 = e_2,\ \ e_i f_1 = \beta_i f_1,\ i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$\begin{gathered} J(e_2,e_2,e_2,f_1)=0 \ \Rightarrow\ \beta_2=0, \\ J(e_1,e_1,e_1,f_1)=0 \ \Rightarrow\ (\beta_1-1)\beta_1(2\beta_1-1)=0, \\ J(e_3,e_3,e_3,f_1)=0 \ \Rightarrow\ (\beta_3-1)\beta_3(2\beta_3-1)=0, \\ J(e_1,e_1,e_3,f_1)=0 \ \Rightarrow\ \beta_1\beta_3(2\beta_1-1)=0, \\ J(e_1,e_3,e_3,f_1)=0 \ \Rightarrow\ \beta_1\beta_3(2\beta_3-1)=0. \end{gathered}$$

- If $\beta_1=0$, then $\beta_3\in\{0,1,\frac{1}{2}\}$, which gives $\mathfrak{J}_{12},\ \mathfrak{J}_{13}$ and $\mathfrak{J}_{14}$.
- If $\beta_1=1$, then $\beta_3=0$, which gives us the superalgebra $\mathcal{J}_{15}$.
- If $\beta_1=\frac{1}{2}$, then $\beta_3\in\{0,\frac{1}{2}\}$, which gives us $\mathfrak{J}_{16}$ and $\mathfrak{J}_{17}$.

Let $\mathfrak{J}_0\cong\mathcal{B}_1\oplus\mathcal{U}_2$. Here we are looking for Jordan superalgebras such that $\mathfrak{J}=(\mathbb{F}e_1+\mathbb{F}e_2+\mathbb{F}e_3)+\mathbb{F}f_1$ with multiplication rules

$$e_1^2=e_1,\quad e_1e_2=e_2,\quad e_if_1=\beta_if_1,\ i=\overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$\begin{gathered} J(e_2,e_2,e_2,f_1)=0 \ \Rightarrow\ \beta_2=0,\ \ J(f_1,f_1,f_1,e_3)=0 \ \Rightarrow\ \beta_3=0, \\ J(e_1,e_1,e_1,f_1)=0 \ \Rightarrow\ (\beta_1-1)\beta_1(2\beta_1-1)=0. \end{gathered}$$

So, we have three superalgebras with $\beta_1\in\{0,1,\frac{1}{2}\}$, which give us $\mathfrak{J}_{18},\ \mathfrak{J}_{19}$ and $\mathfrak{J}_{20}$.

Let $\mathfrak{J}_0\cong\mathcal{B}_2\oplus\mathcal{U}_1$. Here we are looking for Jordan superalgebras such that $\mathfrak{J}=(\mathbb{F}e_1+\mathbb{F}e_2+\mathbb{F}e_3)+\mathbb{F}f_1$ with multiplication rules

$$e_1^2=e_1,\ \ e_3^2=e_3,\ \ e_1e_2=\tfrac{1}{2}e_2,\ \ e_if_1=\beta_if_1,\ i=\overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$\begin{gathered} J(e_2,e_2,e_2,f_1)=0 \ \Rightarrow\ \beta_2=0, \\ J(e_1,e_1,e_1,f_1)=0 \ \Rightarrow\ (\beta_1-1)\beta_1(2\beta_1-1)=0, \\ J(e_3,e_3,e_3,f_1)=0 \ \Rightarrow\ (\beta_3-1)\beta_3(2\beta_3-1)=0, \\ J(e_1,e_1,e_3,f_1)=0 \ \Rightarrow\ \beta_1\beta_3(2\beta_1-1)=0, \\ J(e_1,e_3,e_3,f_1)=0 \ \Rightarrow\ \beta_1\beta_3(2\beta_3-1)=0. \end{gathered}$$

- If $\beta_1=0$, then $\beta_3\in\{0,1,\frac{1}{2}\}$ which gives us $\mathfrak{J}_{21},\ \mathfrak{J}_{22}$ and $\mathfrak{J}_{23}$.
- If $\beta_1=1$, then $\beta_3=0$, which gives us the superalgebra $\mathfrak{J}_{24}$.
- If $\beta_1=\frac{1}{2}$, then $\beta_3\in\{0,\frac{1}{2}\}$, which gives $\mathfrak{J}_{25}$ and $\mathfrak{J}_{26}$.

Let $\mathfrak{J}_0\cong\mathcal{B}_2\oplus\mathcal{U}_2$. Here we are looking for Jordan superalgebras such that $\mathfrak{J}=(\mathbb{F}e_1+\mathbb{F}e_2+\mathbb{F}e_3)+\mathbb{F}f_1$ with multiplication rules

$$e_1^2=e_1,\quad e_1e_2=\tfrac{1}{2}e_2\quad e_if_1=\beta_if_1,\ i=\overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$\begin{gathered} J(e_2,e_2,e_2,f_1)=0 \ \Rightarrow\ \beta_2=0,\ \ J(e_3,e_3,e_3,f_1)=0 \ \Rightarrow\ \beta_3=0, \\ J(e_1,e_1,e_1,f_1)=0 \ \Rightarrow\ (\beta_1-1)\beta_1(2\beta_1-1)=0. \end{gathered}$$

So we have superalgebras $\mathfrak{J}_{27},\ \mathfrak{J}_{28}$ and $\mathfrak{J}_{29}$ from $\beta_1\in\{0,1,\frac{1}{2}\}$.

Let $\mathfrak{J}_0\cong\mathcal{B}_3\oplus\mathcal{U}_1$. Here we are looking for Jordan superalgebras such that $\mathfrak{J}=(\mathbb{F}e_1+\mathbb{F}e_2+\mathbb{F}e_3)+\mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_2, \quad e_3^2 = e_3 \quad e_i f_1 = \beta_i f_1, \ i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_2, e_2, e_2, f_1) = 0 \ \Rightarrow \ \beta_2 = 0, \ \ J(e_1, e_1, e_1, f_1) = 0 \ \Rightarrow \ \beta_1 = 0,$$
$$J(e_3, e_3, e_3, f_1) = 0 \ \Rightarrow \ (\beta_3 - 1)\beta_3(2\beta_3 - 1) = 0.$$

So we have Jordan superalgebras $\mathfrak{J}_{30}, \ \mathfrak{J}_{31}$ and $\mathfrak{J}_{32}$ from $\beta_3 \in \{0, 1, \frac{1}{2}\}$.

Let $\mathfrak{J}_0 \cong \mathcal{B}_3 \oplus \mathcal{U}_2$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_2, \quad e_i f_1 = \beta_i f_1, \ i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_2, e_2, e_2, f_1) = 0 \ \Rightarrow \ \beta_2 = 0, \ \ J(e_1, e_1, e_1, f_1) = 0 \ \Rightarrow \ \beta_1 = 0,$$
$$J(e_3, e_3, e_3, f_1) = 0 \ \Rightarrow \ \beta_3 = 0.$$

So the superalgebra in this case is $\mathfrak{J}_{33}$.

Let $\mathfrak{J}_0 \cong \mathcal{T}_1$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1, \quad e_2^2 = e_3, \quad e_1e_2 = e_2, \quad e_1e_3 = e_3, \quad e_i f_1 = \beta_i f_1, \ i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_3, e_3, e_3, f_1) = 0 \ \Rightarrow \ \beta_3 = 0, \ \ J(e_2, e_2, e_2, f_1) = 0 \ \Rightarrow \ \beta_2 = 0,$$
$$J(e_1, e_1, e_1, f_1) = 0 \ \Rightarrow \ (\beta_1 - 1)\beta_1(2\beta_1 - 1) = 0.$$

Thus, from $\beta_1 \in \{0, 1, \frac{1}{2}\}$, we get $\mathfrak{J}_{34}, \ \mathfrak{J}_{35}$ and $\mathfrak{J}_{36}$.

Let $\mathfrak{J}_0 \cong \mathcal{T}_2$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1, \ \ e_1e_3 = e_3, \ \ e_1e_2 = e_2, \ \ e_i f_1 = \beta_i f_1, \ i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_2, e_2, e_2, f_1) = 0 \ \Rightarrow \ \beta_2 = 0, \ \ J(e_3, e_3, e_3, f_1) = 0 \ \Rightarrow \ \beta_3 = 0,$$
$$J(e_1, e_1, e_1, f_1) = 0 \ \Rightarrow \ (\beta_1 - 1)\beta_1(2\beta_1 - 1) = 0.$$

This we obtain $\mathfrak{J}_{37}, \ \mathfrak{J}_{38}$ and $\mathfrak{J}_{39}$ from $\beta_1 \in \{0, 1, \frac{1}{2}\}$.

Let $\mathfrak{J}_0 \cong \mathcal{T}_3$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_2, \quad e_i f_1 = \beta_i f_1, \ i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_2, e_2, e_2, f_1) = 0 \ \Rightarrow \ \beta_2 = 0, \ \ J(e_3, e_3, e_3, f_1) = 0 \ \Rightarrow \ \beta_3 = 0,$$
$$J(e_1, e_1, e_1, f_1) = 0 \ \Rightarrow \ \beta_1 = 0.$$

Thus we have $\mathfrak{J}_{40}$.

Let $\mathfrak{J}_0 \cong \mathcal{T}_4$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_2, \quad e_1e_3 = e_2, \quad e_if_1 = \beta_if_1, \ i = \overline{1,3}.$$

Using the Jacobi super identity we obtain only the following result:

$$J(e_2,e_2,e_2,f_1) = 0 \ \Rightarrow \ \beta_2 = 0, \ \ J(e_3,e_3,e_3,f_1) = 0 \ \Rightarrow \ \beta_3 = 0,$$
$$J(e_1,e_1,e_1,f_1) = 0 \ \Rightarrow \ \beta_1 = 0.$$

Thus we obtain $\mathfrak{J}_{41}$.

Let $\mathfrak{J}_0 \cong \mathcal{T}_5$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1 \ \ e_3^2 = e_1 + e_2, \ \ e_1e_3 = \tfrac{1}{2}e_3 \ \ e_2^2 = e_2, \ \ e_2e_3 = \tfrac{1}{2}e_3$$
$$e_if_1 = \beta_if_1, \ i = \overline{1,3}.$$

Using the Jacobi super identity we obtain only the following result:

$$J(e_1,e_1,e_1,f_1) = 0 \ \Rightarrow \ (\beta_1 - 1)\beta_1(2\beta_1 - 1) = 0.$$

If $\beta_1 = 0$ then from

$$J(e_1,e_1,e_3,f_1) = 0 \ \Rightarrow \ \beta_3 = 0, \ \ J(e_1,f_1,e_3,e_3) = 0 \ \Rightarrow \ \beta_2 = 0,$$

we get $\mathfrak{J}_{42}$.
If $\beta_1 = 1$ then from

$$J(e_1,e_1,e_2,f_1) = 0 \ \Rightarrow \ \beta_2 = 0, \ \ J(e_1,e_1,e_3,f_1) = 0 \ \Rightarrow \ \beta_3 = 0,$$
$$J(e_1,f_1,e_3,e_3) = 0 \ \Rightarrow \ -\tfrac{1}{2}(1 + \beta_2 - 2\beta_3^2) = 0.$$

we get a contradiction. So there is no superalgebra in this case.
If $\beta_1 = \frac{1}{2}$ then from

$$J(e_1,e_3,e_1,f_1) = 0 \ \Rightarrow \ \beta_3 = 0, \ \ J(e_3,e_3,e_1,f_1) = 0 \ \Rightarrow \ \beta_2 = \tfrac{1}{2}.$$

we have $\mathfrak{J}_{43}$.

Let $\mathfrak{J}_0 \cong \mathcal{T}_6$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (e_1\mathbb{F} + e_2\mathbb{F} + e_3\mathbb{F}) + f_1\mathbb{F}$ with multiplication rules

$$e_1^2 = e_1, \ \ e_1e_2 = \tfrac{1}{2}e_2, \ \ e_1e_3 = e_3, \ \ e_if_1 = \beta_if_1, \ i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_1,e_1,e_1,f_1) = 0 \ \Rightarrow \ (\beta_1 - 1)\beta_1(2\beta_1 - 1) = 0,$$
$$J(e_2,e_2,e_2,f_1) = 0 \ \Rightarrow \ \beta_2 = 0, \ \ J(f_1,e_3,e_3,e_3) = 0 \ \Rightarrow \ \beta_3 = 0.$$

Thus we get $\mathfrak{J}_{44}$, $\mathfrak{J}_{45}$ and $\mathfrak{J}_{46}$ from $\beta_1 = 0, \beta_1 = 1$ and $\beta_1 = \frac{1}{2}$, respectively.

Let $\mathfrak{J}_0 \cong \mathcal{T}_7$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1, \ \ e_1e_2 = \tfrac{1}{2}e_2, \ \ e_1e_3 = \tfrac{1}{2}e_3, \ \ e_if_1 = \beta_if_1 \ i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$\begin{aligned} J(e_2, e_2, e_2, f_1) = 0 \;&\Rightarrow\; \beta_2 = 0, \\ J(e_3, e_3, e_3, f_1) = 0 \;&\Rightarrow\; \beta_3 = 0, \\ J(e_1, e_1, e_1, f_1) = 0 \;&\Rightarrow\; (\beta_1 - 1)\beta_1(2\beta_1 - 1) = 0. \end{aligned}$$

So, we get $\mathfrak{J}_{47}$, $\mathfrak{J}_{48}$ and $\mathfrak{J}_{49}$ superalgebras from $\beta_1 = 0, \beta_1 = 1$ and $\beta_1 = \frac{1}{2}$, respectively.

Let $\mathfrak{J}_0 \cong \mathcal{T}_8$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1, \;\; e_1 e_2 = \tfrac{1}{2} e_2, \;\; e_2^2 = e_3, \;\; e_i f_1 = \beta_i f_1, \;\; i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_3, e_3, e_3, f_1) = 0 \;\Rightarrow\; \beta_3 = 0, \;\; J(e_2, e_2, e_2, f_1) = 0 \;\Rightarrow\; \beta_2 = 0,$$
$$J(e_1, e_1, e_1, f_1) = 0 \;\Rightarrow\; (\beta_1 - 1)\beta_1(2\beta_1 - 1) = 0.$$

Thus we get $\mathfrak{J}_{50}$, $\mathfrak{J}_{51}$ and $\mathfrak{J}_{52}$.

Let $\mathfrak{J}_0 \cong \mathcal{T}_9$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1, \quad e_1 e_2 = \tfrac{1}{2} e_2, \quad e_1 e_3 = e_3, \quad e_2^2 = e_3, \quad e_i f_1 = \beta_i f_1, \;\; i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_3, e_3, e_3, f_1) = 0 \;\Rightarrow\; \beta_3 = 0, \;\; J(e_2, e_2, e_2, f_1) = 0 \;\Rightarrow\; \beta_2 = 0,$$
$$J(e_1, e_1, e_1, f_1) = 0 \;\Rightarrow\; (\beta_1 - 1)\beta_1(2\beta_1 - 1) = 0.$$

Thus we obtain $\mathfrak{J}_{53}$, $\mathfrak{J}_{54}$ and $\mathfrak{J}_{55}$

Let $\mathfrak{J}_0 \cong \mathcal{T}_{10}$. Here we are looking for Jordan superalgebras such that $\mathfrak{J} = (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3) + \mathbb{F}f_1$ with multiplication rules

$$e_1^2 = e_1, \;\; e_1 e_3 = \tfrac{1}{2} e_3, \;\; e_2^2 = e_2, \;\; e_2 e_3 = \tfrac{1}{2} e_3, \;\; e_i f_1 = \beta_i f_1, \;\; i = \overline{1,3}.$$

Using the Jacobi super identity we obtain the following results:

$$J(e_3, e_3, e_3, f_1) = 0 \;\Rightarrow\; \beta_3 = 0,$$
$$J(e_1, e_1, e_1, f_1) = 0 \;\Rightarrow\; (\beta_1 - 1)\beta_1(2\beta_1 - 1) = 0.$$

- If $\beta_1 = 0$ then
$$J(e_2, e_2, e_2, f_1) = 0 \;\Rightarrow\; (\beta_2 - 1)\beta_2(2\beta_2 - 1) = 0.$$
So we have three superalgebras isomorphic to $\mathfrak{J}_{56}$, $\mathfrak{J}_{57}$ and $\mathfrak{J}_{58}$.
- If $\beta_1 = 1$ then $J(e_1, e_1, e_2, f_1) = 0 \;\Rightarrow\; \beta_2 = 0$ gives us $\mathfrak{J}_{57}$.
- If $\beta_1 = \frac{1}{2}$ then $J(e_1, e_2, e_2, f_1) = 0 \;\Rightarrow\; \frac{1}{2}\beta_2(2\beta_2 - 1) = 0$. So we have $\mathfrak{J}_{58}$ and $\mathfrak{J}_{59}$.

□

3 Irreducible components

Theorem 12. *The variety of four-dimensional Jordan superalgebras of type* $(1,3)$ *has dimension* 7 *and it has* 11 *irreducible components defined by*

$$\mathcal{C}_1=\overline{\mathcal{O}(\mathbf{J}_5)},\ \mathcal{C}_2=\overline{\mathcal{O}(\mathbf{J}_7)},\ \mathcal{C}_3=\overline{\mathcal{O}(\mathbf{J}_8)},\ \mathcal{C}_4=\overline{\mathcal{O}(\mathbf{J}_9)},\ \mathcal{C}_5=\overline{\mathcal{O}(\mathbf{J}_{11})},$$
$$\mathcal{C}_6=\overline{\mathcal{O}(\mathbf{J}_{12})},\ \mathcal{C}_7=\overline{\mathcal{O}(\mathbf{J}_{14})},\ \mathcal{C}_8=\overline{\mathcal{O}(\mathbf{J}_{15})},\ \mathcal{C}_9=\overline{\mathcal{O}(\mathbf{J}_{16})},\ \mathcal{C}_{10}=\overline{\mathcal{O}(\mathbf{J}_{17})},$$
$$\mathcal{C}_{11}=\overline{\mathcal{O}(\mathbf{J}_{19})},$$

In particular, all of them are rigid superalgebras.

Proof. Calculating the dimensions of orbit closures of the more important for us superalgebras, we have

$$\begin{aligned}
\dim\mathcal{O}(\mathbf{J}_5)=\dim\mathcal{O}(\mathbf{J}_{12}) &= 7,\\
\dim\mathcal{O}(\mathbf{J}_{11})=\dim\mathcal{O}(\mathbf{J}_{14}) &= 6,\\
\dim\mathcal{O}(\mathbf{J}_8)=\dim\mathcal{O}(\mathbf{J}_9)=\dim\mathcal{O}(\mathbf{J}_{16})=\dim\mathcal{O}(\mathbf{J}_{17}) &= 5,\\
\dim\mathcal{O}(\mathbf{J}_{19}) &= 4,\\
\dim\mathcal{O}(\mathbf{J}_7)=\dim\mathcal{O}(\mathbf{J}_{15}) &= 1.
\end{aligned}$$

If $E_{f_1}^t, E_{f_2}^t, E_{f_3}^t, E_e^t$ is a *parametric basis* for $\mathbf{A}\to\mathbf{B}$, then we denote a degeneration by $\mathbf{A}\xrightarrow{(E_{f_1}^t,E_{f_2}^t,E_{f_3}^t,E_e^t)}\mathbf{B}$.

$\mathbf{J}_3$	$\xrightarrow{(tf_1,f_2,f_3,te)}$	$\mathbf{J}_1$	$\mathbf{J}_5$	$\xrightarrow{(tf_1,tf_2,f_3,e)}$	$\mathbf{J}_2$
$\mathbf{J}_5$	$\xrightarrow{(f_1-f_3,f_2,tf_3,e)}$	$\mathbf{J}_3$	$\mathbf{J}_5$	$\xrightarrow{(tf_1+f_2,tf_2,f_3,e)}$	$\mathbf{J}_4$
$\mathbf{J}_{12}$	$\xrightarrow{(f_1+2f_2+2f_3,tf_2+2tf_3,t^2f_3,te)}$	$\mathbf{J}_6$	$\mathbf{J}_{14}$	$\xrightarrow{(f_1,f_2,tf_3,e)}$	$\mathbf{J}_{10}$
$\mathbf{J}_{14}$	$\xrightarrow{(f_1,f_2,tf_3,e)}$	$\mathbf{J}_{13}$	$\mathbf{J}_{19}$	$\xrightarrow{(tf_1,f_2,f_3,e)}$	$\mathbf{J}_{18}$

Below we list all important reasons for necessary non-degenerations.

Non-degenerations reasons			
$\mathbf{J}_5$	$\not\to$	$\mathbf{J}_7,\mathbf{J}_8,\mathbf{J}_9,\mathbf{J}_{11},\mathbf{J}_{14},\mathbf{J}_{15},\mathbf{J}_{16},\mathbf{J}_{17},\mathbf{J}_{19}$	$\mathcal{R}=\left\{\text{ According to Lemma 5 (2) }\right\}$
$\mathbf{J}_{12}$	$\not\to$	$\mathbf{J}_7,\mathbf{J}_8,\mathbf{J}_9,\mathbf{J}_{11},\mathbf{J}_{14},\mathbf{J}_{15},\mathbf{J}_{16},\mathbf{J}_{17},\mathbf{J}_{19}$	$\mathcal{R}=\left\{ JJ\subset\{e,f_2,f_3\},\ c_{44}^4=2c_{24}^2,\ c_{44}^4=c_{34}^3 \right\}$
$\mathbf{J}_{11}$	$\not\to$	$\mathbf{J}_7,\mathbf{J}_8,\mathbf{J}_9,\mathbf{J}_{15},\mathbf{J}_{16},\mathbf{J}_{17},\mathbf{J}_{19}$	$\mathcal{R}=\left\{ JJ\subset\{e,f_2,f_3\},\ c_{44}^4=2c_{24}^2,\ c_{44}^4=2c_{34}^3,\ c_{24}^3=0 \right\}$
$\mathbf{J}_{14}$	$\not\to$	$\mathbf{J}_7,\mathbf{J}_8,\mathbf{J}_9,\mathbf{J}_{15},\mathbf{J}_{16},\mathbf{J}_{17},\mathbf{J}_{19}$	$\mathcal{R}=\left\{ JJ\subset\{e,f_2,f_3\},\ c_{44}^4=c_{34}^3,\ c_{42}^2=c_{34}^3 \right\}$
$\mathbf{J}_8$	$\not\to$	$\mathbf{J}_7,\mathbf{J}_{15},\mathbf{J}_{19}$	$\mathcal{R}=\left\{ c_{ij}^1=0,\ 2c_{34}^3=c_{44}^4 \right\}$
$\mathbf{J}_9$	$\not\to$	$\mathbf{J}_7,\mathbf{J}_{15},\mathbf{J}_{19}$	$\mathcal{R}=\left\{ c_{ij}^1=0,\ c_{34}^3=c_{44}^4 \right\}$
$\mathbf{J}_{16}$	$\not\to$	$\mathbf{J}_7,\mathbf{J}_{15},\mathbf{J}_{19}$	$\mathcal{R}=\left\{ A_2A_2\subset A_2,\ 2c_{14}^1=c_{44}^4,\ 2c_{24}^2=c_{44}^4,\ c_{34}^3=c_{44}^4 \right\}$
$\mathbf{J}_{17}$	$\not\to$	$\mathbf{J}_7,\mathbf{J}_{15},\mathbf{J}_{19}$	$\mathcal{R}=\left\{ A_2A_2\subset A_2,\ 2c_{14}^1=c_{44}^4,\ c_{24}^2=c_{44}^4,\ c_{34}^3=c_{44}^4 \right\}$
$\mathbf{J}_{19}$	$\not\to$	$\mathbf{J}_7,\mathbf{J}_{15}$	$\mathcal{R}=\left\{ A_2A_2\subset A_2,\ c_{14}^1=c_{44}^4,\ c_{24}^2=c_{44}^4,\ c_{34}^3=c_{44}^4 \right\}$

Here c_{ij}^k coefficients are structural constants in the $x_1 = f_1,\ x_2 = f_2,\ x_3 = f_3,\ x_4 = e$ basis.

□

Theorem 13. *The variety of four-dimensional Jordan superalgebras of type* $(2,2)$ *has dimension* 6 *and it has* 25 *irreducible components defined by*

$$\begin{gathered}
\mathcal{C}_1 = \overline{\mathcal{O}(\mathcal{J}_1)},\ \mathcal{C}_2 = \overline{\mathcal{O}(\mathcal{J}_2)},\ \mathcal{C}_3 = \overline{\mathcal{O}(\mathcal{J}_3)},\ \mathcal{C}_4 = \overline{\mathcal{O}(\mathcal{J}_5)},\ \mathcal{C}_5 = \overline{\mathcal{O}(\mathcal{J}_6)},\\
\mathcal{C}_6 = \overline{\mathcal{O}(\mathcal{J}_8)},\ \mathcal{C}_7 = \overline{\mathcal{O}(\mathcal{J}_9)},\ \mathcal{C}_8 = \overline{\mathcal{O}(\mathcal{J}_{10})},\ \mathcal{C}_9 = \overline{\mathcal{O}(\mathcal{J}_{11})},\ \mathcal{C}_{10} = \overline{\mathcal{O}(\mathcal{J}_{12})},\\
\mathcal{C}_{11} = \overline{\mathcal{O}(\mathcal{J}_{13})},\ \mathcal{C}_{12} = \overline{\mathcal{O}(\mathcal{J}_{14})},\ \mathcal{C}_{13} = \overline{\mathcal{O}(\mathcal{J}_{16}^t)},\ \mathcal{C}_{14} = \overline{\mathcal{O}(\mathcal{J}_{24})},\\
\mathcal{C}_{15} = \overline{\mathcal{O}(\mathcal{J}_{32})},\ \mathcal{C}_{16} = \overline{\mathcal{O}(\mathcal{J}_{42})},\ \mathcal{C}_{17} = \overline{\mathcal{O}(\mathcal{J}_{49})},\ \mathcal{C}_{18} = \overline{\mathcal{O}(\mathcal{J}_{50})},\\
\mathcal{C}_{19} = \overline{\mathcal{O}(\mathcal{J}_{54})},\ \mathcal{C}_{20} = \overline{\mathcal{O}(\mathcal{J}_{56})},\ \mathcal{C}_{21} = \overline{\mathcal{O}(\mathcal{J}_{57})},\ \mathcal{C}_{22} = \overline{\mathcal{O}(\mathcal{J}_{58})},\\
\mathcal{C}_{23} = \overline{\mathcal{O}(\mathcal{J}_{62})},\ \mathcal{C}_{24} = \overline{\mathcal{O}(\mathcal{J}_{64})},\ \mathcal{C}_{25} = \overline{\mathcal{O}(\mathcal{J}_{65})}.
\end{gathered}$$

In particular, 24 of them are rigid superalgebras.

Proof. After carefully checking the dimensions of orbit closures of the more important for us superalgebras, we have

$$\begin{aligned}
\dim\mathcal{O}(\mathcal{J}_2) = \dim\mathcal{O}(\mathcal{J}_3) = \dim\mathcal{O}(\mathcal{J}_5) = \dim\mathcal{O}(\mathcal{J}_9) = \dim\mathcal{O}(\mathcal{J}_{10}))\\
= \dim\mathcal{O}(\mathcal{J}_{11} = \dim\mathcal{O}(\mathcal{J}_{12}) = \dim\mathcal{O}(\mathcal{J}_{13}) = \dim\mathcal{O}(\mathcal{J}_{14}) = \dim\mathcal{O}(\mathcal{J}_{16}^t)\\
= \dim\mathcal{O}(\mathcal{J}_{24}) = \dim\mathcal{O}(\mathcal{J}_{32}) = \dim\mathcal{O}(\mathcal{J}_{42}) = \dim\mathcal{O}(\mathcal{J}_{49}) = \dim\mathcal{O}(\mathcal{J}_{54})\\
= \dim\mathcal{O}(\mathcal{J}_{56}) = \dim\mathcal{O}(\mathcal{J}_{62}) = \dim\mathcal{O}(\mathcal{J}_{64}) = 6,\\
\dim\mathcal{O}(\mathcal{J}_6) = \dim\mathcal{O}(\mathcal{J}_8) = 5,\\
\dim\mathcal{O}(\mathcal{J}_1) = \dim\mathcal{O}(\mathcal{J}_{57}) = 4,\\
\dim\mathcal{O}(\mathcal{J}_{50}) = \dim\mathcal{O}(\mathcal{J}_{58}) = \dim\mathcal{O}(\mathcal{J}_{65}) = 2.
\end{aligned}$$

If $E_{e_1}^t, E_{e_2}^t, E_{f_1}^t, E_{f_2}^t$ is a *parametric basis* for $\mathbf{A} \to \mathbf{B}$, then we denote a degeneration by $\mathbf{A} \xrightarrow{(E_{e_1}^t, E_{e_2}^t, E_{f_1}^t, E_{f_2}^t)} \mathbf{B}$.

$\mathcal{J}_{6}$	$\xrightarrow{(e_1,e_2,tf_1,f_2)}$	$\mathcal{J}_{4}$	$\mathcal{J}_{8}$	$\xrightarrow{(e_1,e_2,tf_1,f_2)}$	$\mathcal{J}_{7}$	$\mathcal{J}_{16}^{0}$	$\xrightarrow{(e_1,e_2,f_1,tf_2)}$	$\mathcal{J}_{15}$
$\mathcal{J}_{68}$	$\xrightarrow{(e_1,\frac{1}{t}e_2,f_1,f_2)}$	$\mathcal{J}_{17}$	$\mathcal{J}_{23}$	$\xrightarrow{(te_1,e_2,f_1,f_2)}$	$\mathcal{J}_{18}$	$\mathcal{J}_{24}$	$\xrightarrow{(te_1,e_2,f_1,f_2)}$	$\mathcal{J}_{20}$
$\mathcal{J}_{22}$	$\xrightarrow{(e_1,e_2,tf_1,f_2)}$	$\mathcal{J}_{21}$	$\mathcal{J}_{24}$	$\xrightarrow{(e_1,te_2,tf_1,f_2)}$	$\mathcal{J}_{22}$	$\mathcal{J}_{24}$	$\xrightarrow{(e_1,e_2,tf_1,tf_2)}$	$\mathcal{J}_{23}$
$\mathcal{J}_{13}$	$\xrightarrow{(e_1,te_2,f_1,f_2)}$	$\mathcal{J}_{25}$	$\mathcal{J}_{14}$	$\xrightarrow{(e_1,te_2,f_1,f_2)}$	$\mathcal{J}_{26}$	$\mathcal{J}_{28}$	$\xrightarrow{(e_1,e_2,tf_1,f_2)}$	$\mathcal{J}_{27}$
$\mathcal{J}_{30}$	$\xrightarrow{(e_1,te_2,f_1,tf_2)}$	$\mathcal{J}_{28}$	$\mathcal{J}_{30}$	$\xrightarrow{(e_1,\frac{1}{t}e_2,f_1,f_2)}$	$\mathcal{J}_{29}$	$\mathcal{J}_{16}^{t}$	$\xrightarrow{(e_1,te_2,f_1,f_2)}$	$\mathcal{J}_{30}$
$\mathcal{J}_{32}$	$\xrightarrow{(e_1,\frac{1}{t}e_2,tf_1,f_2)}$	$\mathcal{J}_{31}$	$\mathcal{J}_{35}$	$\xrightarrow{(e_1,e_2,f_1,tf_2)}$	$\mathcal{J}_{34}$	$\mathcal{J}_{1}$	$\xrightarrow{(e_1+e_2,te_2,f_1,f_2)}$	$\mathcal{J}_{36}$
$\mathcal{J}_{40}$	$\xrightarrow{(e_1,e_2,f_1,tf_2)}$	$\mathcal{J}_{39}$	$\mathcal{J}_{42}$	$\xrightarrow{(e_1,te_2,tf_1,f_2)}$	$\mathcal{J}_{40}$	$\mathcal{J}_{42}$	$\xrightarrow{(e_1,e_2,tf_1,tf_2)}$	$\mathcal{J}_{41}$
$\mathcal{J}_{5}$	$\xrightarrow{(e_1+e_2,te_2,f_2,f_1)}$	$\mathcal{J}_{43}$	$\mathcal{J}_{45}$	$\xrightarrow{(e_1,e_2,tf_1,f_2)}$	$\mathcal{J}_{44}$	$\mathcal{J}_{47}$	$\xrightarrow{(e_1,te_2,f_1,tf_2)}$	$\mathcal{J}_{45}$
$\mathcal{J}_{47}$	$\xrightarrow{(e_1,\frac{1}{t}e_2,f_1,f_2)}$	$\mathcal{J}_{46}$	$\mathcal{J}_{49}$	$\xrightarrow{(e_1,e_2,tf_1,tf_2)}$	$\mathcal{J}_{48}$	$\mathcal{J}_{52}$	$\xrightarrow{(e_1,e_2,f_1,tf_2)}$	$\mathcal{J}_{51}$
$\mathcal{J}_{54}$	$\xrightarrow{(e_1,te_2,f_1,tf_2)}$	$\mathcal{J}_{52}$	$\mathcal{J}_{54}$	$\xrightarrow{(e_1,e_2,tf_1,tf_2)}$	$\mathcal{J}_{53}$	$\mathcal{J}_{56}$	$\xrightarrow{(e_1,e_2,tf_1,tf_2)}$	$\mathcal{J}_{55}$
$\mathcal{J}_{60}$	$\xrightarrow{(e_1,e_2,f_1,tf_2)}$	$\mathcal{J}_{59}$	$\mathcal{J}_{64}$	$\xrightarrow{(e_1,te_2,tf_1,f_2)}$	$\mathcal{J}_{60}$	$\mathcal{J}_{62}$	$\xrightarrow{(e_1,e_2,tf_1,tf_2)}$	$\mathcal{J}_{61}$
$\mathcal{J}_{64}$	$\xrightarrow{(e_1,e_2,tf_1,tf_2)}$	$\mathcal{J}_{63}$	$\mathcal{J}_{67}$	$\xrightarrow{(e_1,e_2,f_1,tf_2)}$	$\mathcal{J}_{66}$	$\mathcal{J}_{71}$	$\xrightarrow{(e_1,e_2,tf_1,tf_2)}$	$\mathcal{J}_{70}$

$\mathcal{J}_{\mathbf{42}}$	$\xrightarrow{(-t^5e_1,t^3e_1+e_2,tf_1,tf_2)}$	$\mathcal{J}_{\mathbf{19}}$
$\mathcal{J}_{\mathbf{5}}$	$\xrightarrow{(\frac{t}{3}e_1+(1+\frac{2t}{3})e_2,\sqrt{t}e_1+\sqrt{t}e_2,\sqrt{t}f_1+(2t-3)f_2,f_1+4\sqrt{t}f_2)}$	$\mathcal{J}_{\mathbf{33}}$
$\mathcal{J}_{\mathbf{6}}$	$\xrightarrow{(-te_1+(1+t)e_2,\sqrt{t}e_1+\sqrt{t}e_2,f_1,(1+2t)f_2)}$	$\mathcal{J}_{\mathbf{35}}$
$\mathcal{J}_{\mathbf{3}}$	$\xrightarrow{((1-t)e_1+(1+t)e_2,\frac{(t-1)\sqrt{t}}{1+t}e_1+\sqrt{t}e_2,f_1,f_2)}$	$\mathcal{J}_{\mathbf{37}}$
$\mathcal{J}_{\mathbf{2}}$	$\xrightarrow{((1+2t-t^2)e_1+(1+t^2)e_2,\frac{\sqrt{t}(-1-2t+t^2)}{1+t^2}e_1+\sqrt{t}e_2,f_1,f_2)}$	$\mathcal{J}_{\mathbf{38}}$
$\mathcal{J}_{\mathbf{16}}^{\mathbf{1+t}}$	$\xrightarrow{(e_1+e_2,te_2,f_1,f_2)}$	$\mathcal{J}_{\mathbf{47}}$
$\mathcal{J}_{\mathbf{68}}$	$\xrightarrow{(te_1+e_2,t^2e_2,-t^3f_1,f_2)}$	$\mathcal{J}_{\mathbf{67}}$
$\mathcal{J}_{\mathbf{16}}^{\mathbf{-1}}$	$\xrightarrow{(te_1-te_2,\frac{2t^2}{1+t}e_2,f_1,tf_2)}$	$\mathcal{J}_{\mathbf{68}}$
$\mathcal{J}_{\mathbf{9}}$	$\xrightarrow{(-\frac{t}{\sqrt{2}}e_1+\frac{t}{\sqrt{2}}e_2,t^2e_2,f_1+2f_2,t^2f_2)}$	$\mathcal{J}_{\mathbf{69}}$
$\mathcal{J}_{\mathbf{24}}$	$\xrightarrow{(-t^2e_1+e_2,t^2e_2,tf_1,tf_2)}$	$\mathcal{J}_{\mathbf{71}}$

Below we list all important reasons for necessary non-degenerations. All other nondegenerations which are not in this table, can be inferred from Theorem 7 and Lemma 5(2). Since the even parts of $\mathcal{J}_{56}, \mathcal{J}_{54}, \mathcal{J}_{57}, \mathcal{J}_{64}, \mathcal{J}_{50}, \mathcal{J}_{65}, \mathcal{J}_{58}$ superalgebras coincide with $\mathcal{B}_2$, we conclude that these superalgebras do not degenerate to others whose even part is not isomorphic to $\mathcal{B}_2$ and vice versa.

Non-degenerations reasons			
$\mathcal{J}_{\mathbf{2}}$	$\not\to$	$\mathcal{J}_{\mathbf{1}}, \mathcal{J}_{\mathbf{6}}, \mathcal{J}_{\mathbf{8}}$	$\mathcal{R} = \left\{ \begin{array}{l} A_1A_3 \subset A_4,\ c_{12}^2 = c_{14}^4,\ c_{24}^4 = c_{22}^2, \\ c_{11}^2c_{23}^4 = c_{13}^4(c_{14}^4 - c_{11}^1) \end{array} \right\}$
$\mathcal{J}_{\mathbf{3}}$	$\not\to$	$\mathcal{J}_{\mathbf{1}}, \mathcal{J}_{\mathbf{6}}, \mathcal{J}_{\mathbf{8}}$	$\mathcal{R} = \left\{ \begin{array}{l} A_1A_3 \subset A_4,\ c_{12}^2 = 2c_{14}^4,\ 2c_{24}^4 = c_{22}^2, \\ c_{11}^2c_{23}^4 = c_{13}^4(2c_{14}^4 - c_{11}^1) \end{array} \right\}$
$\mathcal{J}_{\mathbf{5}}$	$\not\to$	$\mathcal{J}_{\mathbf{1}}, \mathcal{J}_{\mathbf{6}}, \mathcal{J}_{\mathbf{8}}$	$\mathcal{R} = \left\{ c_{22}^1 = 0,\ c_{23}^3 = c_{22}^2,\ c_{34}^1 = 0,\ c_{34}^2 = 0 \right\}$
$\mathcal{J}_{\mathbf{9}}$	$\not\to$	$\mathcal{J}_{\mathbf{1}}, \mathcal{J}_{\mathbf{6}}, \mathcal{J}_{\mathbf{8}}$	$\mathcal{R} = \left\{ A_1A_3 \subset A_4,\ c_{12}^1 = 0,\ 2c_{24}^4 = c_{22}^2,\ 2c_{14}^4 = c_{11}^1 + c_{12}^2 \right\}$
$\mathcal{J}_{\mathbf{10}}$	$\not\to$	$\mathcal{J}_{\mathbf{1}}, \mathcal{J}_{\mathbf{6}}, \mathcal{J}_{\mathbf{8}}$	$\mathcal{R} = \left\{ c_{22}^1 = 0,\ c_{23}^3 = c_{22}^2,\ c_{34}^1 = 0,\ c_{34}^2 = 0 \right\}$
$\mathcal{J}_{\mathbf{11}}$	$\not\to$	$\mathcal{J}_{\mathbf{1}}, \mathcal{J}_{\mathbf{6}}, \mathcal{J}_{\mathbf{8}}$	$\mathcal{R} = \left\{ c_{22}^1 = 0,\ c_{23}^3 = c_{22}^2,\ c_{34}^1 = 0,\ c_{34}^2 = 0 \right\}$
$\mathcal{J}_{\mathbf{12}}$	$\not\to$	$\mathcal{J}_{\mathbf{1}}, \mathcal{J}_{\mathbf{6}}, \mathcal{J}_{\mathbf{8}}$	$\mathcal{R} = \left\{ c_{22}^1 = 0,\ c_{22}^2 = 2c_{23}^3,\ c_{34}^1 = 0,\ c_{34}^2 = 0 \right\}$
$\mathcal{J}_{\mathbf{13}}$	$\not\to$	$\mathcal{J}_{\mathbf{1}}, \mathcal{J}_{\mathbf{6}}, \mathcal{J}_{\mathbf{8}}$	$\mathcal{R} = \left\{ c_{22}^1 = 0,\ c_{34}^1 = 0,\ c_{34}^2 = 0,\ 2c_{23}^3 = c_{22}^2 \right\}$
$\mathcal{J}_{\mathbf{14}}$	$\not\to$	$\mathcal{J}_{\mathbf{1}}, \mathcal{J}_{\mathbf{6}}, \mathcal{J}_{\mathbf{8}}$	$\mathcal{R} = \left\{ c_{22}^1 = 0,\ c_{34}^1 = 0,\ c_{34}^2 = 0,\ c_{23}^3 = c_{22}^2 \right\}$
$\mathcal{J}_{\mathbf{16}}^{\mathbf{t}}$	$\not\to$	$\mathcal{J}_{\mathbf{1}}, \mathcal{J}_{\mathbf{6}}, \mathcal{J}_{\mathbf{8}}$	$\mathcal{R} = \left\{ c_{12}^1 = 0,\ c_{22}^1 = 0,\ c_{22}^2 = 2c_{24}^4,\ c_{11}^1 + c_{12}^2 = 2c_{14}^4 \right\}$
$\mathcal{J}_{\mathbf{54}}$	$\not\to$	$\mathcal{J}_{\mathbf{50}}, \mathcal{J}_{\mathbf{57}}, \mathcal{J}_{\mathbf{58}}, \mathcal{J}_{\mathbf{65}}$	$\mathcal{R} = \left\{ c_{12}^1 = 0,\ c_{11}^1 - c_{12}^2 - c_{13}^3 - c_{14}^4 = 0 \right\}$
$\mathcal{J}_{\mathbf{56}}$	$\not\to$	$\mathcal{J}_{\mathbf{50}}, \mathcal{J}_{\mathbf{57}}, \mathcal{J}_{\mathbf{58}}, \mathcal{J}_{\mathbf{65}}$	$\mathcal{R} = \left\{ c_{12}^1 = 0,\ c_{13}^3 = 0,\ c_{11}^1 = 2c_{14}^4, \right\}$
$\mathcal{J}_{\mathbf{62}}$	$\not\to$	$\mathcal{J}_{\mathbf{50}}, \mathcal{J}_{\mathbf{57}}, \mathcal{J}_{\mathbf{58}}, \mathcal{J}_{\mathbf{65}}$	$\mathcal{R} = \left\{ c_{12}^1 = 0,\ 2c_{11}^1 - c_{12}^2 - c_{13}^3 - c_{14}^4 = 0 \right\}$
$\mathcal{J}_{\mathbf{64}}$	$\not\to$	$\mathcal{J}_{\mathbf{50}}, \mathcal{J}_{\mathbf{57}}, \mathcal{J}_{\mathbf{58}}, \mathcal{J}_{\mathbf{65}}$	$\mathcal{R} = \left\{ c_{12}^1 = 0,\ 2c_{11}^1 - c_{12}^2 - c_{13}^3 - c_{14}^4 = 0 \right\}$
$\mathcal{J}_{\mathbf{6}}$	$\not\to$	$\mathcal{J}_{\mathbf{1}}$	$\mathcal{R} = \left\{ c_{22}^1 = 0,\ c_{24}^4 = c_{22}^2 \right\}$
$\mathcal{J}_{\mathbf{8}}$	$\not\to$	$\mathcal{J}_{\mathbf{1}}$	$\mathcal{R} = \left\{ c_{22}^1 = 0,\ 2c_{24}^4 = c_{22}^2 \right\}$
$\mathcal{J}_{\mathbf{57}}$	$\not\to$	$\mathcal{J}_{\mathbf{50}}, \mathcal{J}_{\mathbf{58}}, \mathcal{J}_{\mathbf{65}}$	$\mathcal{R} = \left\{ c_{12}^1 = 0,\ c_{13}^3 = 0,\ c_{11}^1 - c_{14}^4 = 0 \right\}$

Here c_{ij}^{k} coefficients are structural constants in the $x_1 = e_1,\ x_2 = e_2,\ x_3 = f_1,\ x_4 = f_2$ basis.

□

Theorem 14. *The variety of four-dimensional Jordan superalgebras of type* $(3,1)$ *has dimension* 9 *and it has* 21 *irreducible components defined by*

$$\mathcal{C}_1 = \overline{\mathcal{O}(\mathfrak{J}_1)},\ \ \mathcal{C}_2 = \overline{\mathcal{O}(\mathfrak{J}_2)},\ \ \mathcal{C}_3 = \overline{\mathcal{O}(\mathfrak{J}_3)},\ \ \mathcal{C}_4 = \overline{\mathcal{O}(\mathfrak{J}_4)},\ \ \mathcal{C}_5 = \overline{\mathcal{O}(\mathfrak{J}_{21})},$$
$$\mathcal{C}_6 = \overline{\mathcal{O}(\mathfrak{J}_{22})},\ \ \mathcal{C}_7 = \overline{\mathcal{O}(\mathfrak{J}_{23})},\ \ \mathcal{C}_8 = \overline{\mathcal{O}(\mathfrak{J}_{24})},\ \ \mathcal{C}_9 = \overline{\mathcal{O}(\mathfrak{J}_{25})},\ \ \mathcal{C}_{10} = \overline{\mathcal{O}(\mathfrak{J}_{26})},$$
$$\mathcal{C}_{11} = \overline{\mathcal{O}(\mathfrak{J}_{42})},\ \ \mathcal{C}_{12} = \overline{\mathcal{O}(\mathfrak{J}_{43})},\ \ \mathcal{C}_{13} = \overline{\mathcal{O}(\mathfrak{J}_{47})},\ \ \mathcal{C}_{14} = \overline{\mathcal{O}(\mathfrak{J}_{48})},\ \ \mathcal{C}_{15} = \overline{\mathcal{O}(\mathfrak{J}_{49})},$$
$$\mathcal{C}_{16} = \overline{\mathcal{O}(\mathfrak{J}_{51})},\ \ \mathcal{C}_{17} = \overline{\mathcal{O}(\mathfrak{J}_{53})},\ \ \mathcal{C}_{18} = \overline{\mathcal{O}(\mathfrak{J}_{54})},\ \ \mathcal{C}_{19} = \overline{\mathcal{O}(\mathfrak{J}_{55})},\ \ \mathcal{C}_{20} = \overline{\mathcal{O}(\mathfrak{J}_{57})},$$
$$\mathcal{C}_{21} = \overline{\mathcal{O}(\mathfrak{J}_{58})}.$$

In particular, all of them are rigid superalgebras.

Proof. After carefully checking the dimensions of orbit closures of the more important for us superalgebras, we have

$$\begin{aligned}
\dim \mathcal{O}(\mathfrak{J}_1) = \dim \mathcal{O}(\mathfrak{J}_2) = \dim \mathcal{O}(\mathfrak{J}_3) = \dim \mathcal{O}(\mathfrak{J}_4) &= 9,\\
\dim \mathcal{O}(\mathfrak{J}_{42}) = \dim \mathcal{O}(\mathfrak{J}_{43}) &= 8,\\
\dim \mathcal{O}(\mathfrak{J}_{21}) = \dim \mathcal{O}(\mathfrak{J}_{22}) = \dim \mathcal{O}(\mathfrak{J}_{23}) = \dim \mathcal{O}(\mathfrak{J}_{24}) &\\
= \dim \mathcal{O}(\mathfrak{J}_{25}) = \dim \mathcal{O}(\mathfrak{J}_{26}) = \dim \mathcal{O}(\mathfrak{J}_{51}) = \dim \mathcal{O}(\mathfrak{J}_{53}) &\\
= \dim \mathcal{O}(\mathfrak{J}_{54}) = \dim \mathcal{O}(\mathfrak{J}_{55}) = \dim \mathcal{O}(\mathfrak{J}_{57}) = \dim \mathcal{O}(\mathfrak{J}_{58}) &= 7,\\
\dim \mathcal{O}(\mathfrak{J}_{47}) = \dim \mathcal{O}(\mathfrak{J}_{48}) = \dim \mathcal{O}(\mathfrak{J}_{49}) &= 3.
\end{aligned}$$

If $E_{e_1}^t, E_{e_2}^t, E_{e_3}^t, E_f^t$ is a *parametric basis* for $\mathbf{A} \to \mathbf{B}$, then we denote a degeneration by $\mathbf{A} \xrightarrow{(E_{f_1}^t, E_{f_2}^t, E_{f_3}^t, E_e^t)} \mathbf{B}$.

$\mathfrak{J}_{1}$	$\xrightarrow{(e_1,e_2,te_3,f)}$	$\mathfrak{J}_{5}$	$\mathfrak{J}_{2}$	$\xrightarrow{(e_1,e_2,te_3,f)}$	$\mathfrak{J}_{6}$
$\mathfrak{J}_{3}$	$\xrightarrow{(e_1,e_2,te_3,f)}$	$\mathfrak{J}_{7}$	$\mathfrak{J}_{4}$	$\xrightarrow{(e_1,e_2,te_3,f)}$	$\mathfrak{J}_{8}$
$\mathfrak{J}_{1}$	$\xrightarrow{(e_1,te_2,te_3,f)}$	$\mathfrak{J}_{9}$	$\mathfrak{J}_{2}$	$\xrightarrow{(e_1,te_2,te_3,f)}$	$\mathfrak{J}_{10}$
$\mathfrak{J}_{3}$	$\xrightarrow{(e_1,te_2,te_3,f)}$	$\mathfrak{J}_{11}$	$\mathfrak{J}_{1}$	$\xrightarrow{(e_1+e_2,te_2,e_3,f)}$	$\mathfrak{J}_{12}$
$\mathfrak{J}_{2}$	$\xrightarrow{(e_2+e_3,te_2,e_1,f)}$	$\mathfrak{J}_{13}$	$\mathfrak{J}_{3}$	$\xrightarrow{(e_2+e_3,te_2,e_1,f)}$	$\mathfrak{J}_{14}$
$\mathfrak{J}_{2}$	$\xrightarrow{(e_1+e_2,te_2,e_3,f)}$	$\mathfrak{J}_{15}$	$\mathfrak{J}_{3}$	$\xrightarrow{(e_1+e_2,te_2,e_3,f)}$	$\mathfrak{J}_{16}$
$\mathfrak{J}_{4}$	$\xrightarrow{(e_2+e_3,te_2,e_1,f)}$	$\mathfrak{J}_{17}$	$\mathfrak{J}_{1}$	$\xrightarrow{(e_1+e_2,te_2,te_3,f)}$	$\mathfrak{J}_{18}$
$\mathfrak{J}_{2}$	$\xrightarrow{(e_1+e_2,te_2,te_3,f)}$	$\mathfrak{J}_{19}$	$\mathfrak{J}_{3}$	$\xrightarrow{(e_1+e_2,te_2,te_3,f)}$	$\mathfrak{J}_{20}$
$\mathfrak{J}_{21}$	$\xrightarrow{(e_1,e_2,te_3,f)}$	$\mathfrak{J}_{27}$	$\mathfrak{J}_{51}$	$\xrightarrow{(e_1,te_2,e_3,f)}$	$\mathfrak{J}_{28}$
$\mathfrak{J}_{26}$	$\xrightarrow{(e_1,e_2,te_3,f)}$	$\mathfrak{J}_{29}$	$\mathfrak{J}_{1}$	$\xrightarrow{((t-t^2)e_1+te_2,t^3e_2,e_3,f)}$	$\mathfrak{J}_{30}$
$\mathfrak{J}_{2}$	$\xrightarrow{(te_2+t^2e_3,t^2e_2+t^4e_3,e_1,f)}$	$\mathfrak{J}_{31}$	$\mathfrak{J}_{3}$	$\xrightarrow{(te_2+t^2e_3,t^2e_2+t^4e_3,e_1,f)}$	$\mathfrak{J}_{32}$
$\mathfrak{J}_{1}$	$\xrightarrow{((t-t^2)e_1+te_2,t^3e_2,te_3,f)}$	$\mathfrak{J}_{33}$	$\mathfrak{J}_{1}$	$\xrightarrow{(e_1+e_2+e_3,(t-t^2)e_2+te_3,t^3e_3,f)}$	$\mathfrak{J}_{34}$
$\mathfrak{J}_{2}$	$\xrightarrow{(e_1+e_2+e_3,(t-t^2)e_2+te_3,t^3e_3,f)}$	$\mathfrak{J}_{35}$	$\mathfrak{J}_{3}$	$\xrightarrow{(e_1+e_2+e_3,(t-t^2)e_2+te_3,t^3e_3,f)}$	$\mathfrak{J}_{36}$
$\mathfrak{J}_{1}$	$\xrightarrow{(e_1+e_2+e_3,te_2,te_3,f)}$	$\mathfrak{J}_{37}$	$\mathfrak{J}_{2}$	$\xrightarrow{(e_1+e_2+e_3,te_2,te_3,f)}$	$\mathfrak{J}_{38}$
$\mathfrak{J}_{3}$	$\xrightarrow{(e_1+e_2+e_3,te_2,te_3,f)}$	$\mathfrak{J}_{39}$	$\mathfrak{J}_{34}$	$\xrightarrow{(te_1+e_2,te_2+e_3,te_3,f)}$	$\mathfrak{J}_{40}$

$\mathfrak{J}_{40}$	$\xrightarrow{(te_1+\frac{1+t}{2}e_2,te_3,e_2+e_3,f)}$	$\mathfrak{J}_{41}$	$\mathfrak{J}_{21}$	$\xrightarrow{(e_1+e_3,e_2,te_3,f)}$	$\mathfrak{J}_{44}$
$\mathfrak{J}_{22}$	$\xrightarrow{(e_1+e_3,e_2,te_3,f)}$	$\mathfrak{J}_{45}$	$\mathfrak{J}_{23}$	$\xrightarrow{(e_1+e_3,e_2,te_3,f)}$	$\mathfrak{J}_{46}$
$\mathfrak{J}_{42}$	$\xrightarrow{(e_1,te_3,t^2e_2,f)}$	$\mathfrak{J}_{50}$	$\mathfrak{J}_{43}$	$\xrightarrow{(e_1,te_3,t^2e_2,f)}$	$\mathfrak{J}_{52}$
$\mathfrak{J}_{42}$	$\xrightarrow{(e_1,e_2,te_3,f)}$	$\mathfrak{J}_{56}$	$\mathfrak{J}_{43}$	$\xrightarrow{(e_1,e_2,te_3,f)}$	$\mathfrak{J}_{59}$

Below we list all important reasons for necessary non-degenerations. All other nondegenerations which are not in this table, can be inferred from Theorem 8 and Lemma 5(2).

Non-degenerations reasons	
$\mathfrak{J}_{42} \not\to \mathfrak{J}_{51}, \mathfrak{J}_{57}, \mathfrak{J}_{58}$	$\mathcal{R} = \left\{ A_1A_4 = 0 \right\}$
$\mathfrak{J}_{43} \not\to \mathfrak{J}_{51}, \mathfrak{J}_{57}, \mathfrak{J}_{58}$	$\mathcal{R} = \left\{ \begin{array}{l} c_{12}^1 + c_{22}^2 + c_{23}^3 = 3c_{24}^4, \\ c_{11}^1 + c_{12}^2 + c_{13}^3 = 3c_{14}^4, \\ c_{34}^4 = 0,\ c_{23}^2 + c_{33}^3 + c_{13}^1 = 0. \end{array} \right\}$

Here c_{ij}^k coefficients are structural constants in the $x_1 = e_1,\ x_2 = e_2,\ x_3 = e_3,\ x_4 = f$ basis.

□

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Kobiljon Abdurasulov
CMA-UBI, University of Beira Interior, Covilhã, Portugal;
Saint Petersburg University, Russia;
Institute of Mathematics Academy of Sciences of Uzbekistan, Tashkent, Uzbekistan
Email address: abdurasulov0505@mail.ru

Roman Lubkov
Department of Mathematics and Computer Science, Saint Petersburg University,
7/9 Universitetskaya nab.,
199034 St. Petersburg, Russia
Email address: r.lubkov@spbu.ru, romanlubkov@yandex.ru

Azamat Saydaliyev
Institute of Mathematics, Academy of Sciences of Uzbekistan, Tashkent, Uzbekistan;
National University of Uzbekistan, Tashkent, Uzbekistan
Email address: azamatsaydaliyev6@gmail.com