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СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Vol. 22, No. 1, pp. 800–812 (2025) УДК 517.9
https://doi.org/10.33048/semi.2025.22.051 MSC 39A12, 39A13, 34L10

EIGENFUNCTION EXPANSIONS OF IMPULSIVE q-STURM–LIOUVILLE PROBLEMS

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Communicated by E.M. Rudoy

Abstract: In this work, an impulsive q-Sturm–Liouville problem is studied. The existence of a countably infinite set of eigenvalues and eigenfunctions is proved and a uniformly convergent expansion formula in the eigenfunctions is established.

Keywords: Difference equations, impulsive conditions, q-Sturm–Liouville problems, eigenfunction expansions.

1 Introduction

The Fourier method is one of the important methods used in the solution of partial differential equations. While solving with this method, the partial differential equation considered is reduced to Sturm–Liouville problems. The eigenvalues and eigenfunctions of these problems are investigated. Meanwhile, the problem of uniform convergence of eigenfunction expansions arises. For this reason, eigenvalues, eigenfunctions, and eigenfunction expansions of Sturm–Liouville problems have been extensively studied in the theory of differential equations (see [1, 2, 3, 4, 6, 9, 10, 11, 12, 14, 15]).

Quantum calculus, which has a long history, has recently started to attract the attention of many researchers (see [5]). In 2005 Annaby and

Allahverdiev, B.P., Tuna, H., Isaev, H.A., Eigenfunction expansions of impulsive q-Sturm–Liouville problems.

Received July, 13, 2023, Published July, 8, 2025.

Mansour studied the q-analogue of the classical Sturm–Liouville equations (see [1]). Classical results are obtained by considering the problem in the regular case. Thereupon, the researchers began to study whether the results obtained in the classical Sturm–Liouville theory were valid for the q-Sturm–Liouville problems. In this context, the existence of countably infinite sets of eigenvalues and eigenvectors of impulsive q-Sturm–Liouville problems and uniform convergence of eigenfunction expansions will be investigated. The Uniform convergence problem will be investigated using Steklov's method (see [2, 6, 13]).

2 Preliminaries

In this section, the basic concepts of q-calculus that will be used in the article will be given. For more detailed information, the following sources can be examined, [8, 1, 5].

Let $q \in (0,1)$ and let $A \subset \mathbb{R}$ be a q-geometric set, i.e., if $q\zeta \in A$ for all $\zeta \in A$. We begin by defining the *operator* $\mathcal{D}_q$ by

$$\mathcal{D}_q f(\zeta) = \begin{cases} \frac{f(q\zeta)-f(\zeta)}{(q-1)\zeta}, & \zeta \neq 0 \\ \lim_{n\to\infty} \frac{f(q^n\xi)-f(0)}{q^n\xi}, & \zeta = 0, \end{cases}$$

where $\zeta, \xi \in A$. We define the *Jackson q-integration* by

$$\int_0^{\zeta} f(\gamma)\, d_q\gamma = \zeta(1-q)\sum_{n=0}^{\infty} q^n f(q^n\zeta),$$

where $\zeta \in A$. From [7], we have

$$\int_0^{\infty} f(\gamma)\, d_q\gamma = \sum_{n=-\infty}^{\infty} q^n f(q^n).$$

Through the remainder of the paper, we deal only with functions q-regular at zero, i.e., functions satisfying

$$\lim_{n\to\infty} f(\zeta q^n) = f(0),$$

for every $\zeta \in A$.

3 Main Results

Consider the following problem

$$(\tau y)(\zeta) := -\frac{1}{q} D_{q^{-1}} D_q y(\zeta) + v(\zeta)\, y(\zeta) = \lambda y(\zeta), \ \zeta \in (0,d) \cup (d,a) \quad (1)$$

with the boundary and impulsive conditions

$$\begin{aligned} y(0) - h_1 D_{q^{-1}} y(0) = 0, \\ y(a) + h_2 D_{q^{-1}} y(a) = 0, \end{aligned} \quad (2)$$

$$y(d-) = \eta y(d+), \tag{3}$$

$$\mathcal{D}_{q^{-1}} y(d-) = \frac{1}{\eta} \mathcal{D}_{q^{-1}} y(d+), \tag{4}$$

where $h_1, h_2, \eta > 0$, λ is a complex eigenvalue parameter, v is a real-valued continuous functions on $[0,d) \cup (d, q^{-1}a]$ and has finite limits $v(d\pm)$, $v(\zeta) \geq 0$, $\zeta \in [0,d) \cup (d, q^{-1}a]$, $a > 0$.

$H = L_q^2(0,d) \dot{+} L_q^2(d,a)$ is a Hilbert space endowed with the following inner product

$$\langle y, z \rangle := \int_0^d y^{(1)} \overline{z^{(1)}} d_q\zeta + \int_d^a y^{(2)} \overline{z^{(2)}} d_q\zeta,$$

where

$$y(\zeta) = \begin{cases} y^{(1)}(\zeta), & \zeta \in (0,d) \\ y^{(2)}(\zeta), & \zeta \in (d,a) \end{cases}$$

and

$$z(\zeta) = \begin{cases} z^{(1)}(\zeta), & \zeta \in (0,d) \\ z^{(2)}(\zeta), & \zeta \in (d,a). \end{cases}$$

Let

$$\mathcal{T} : \mathcal{D} \subset H \to H, \ \mathcal{T}y = \tau y, \ y \in \mathcal{D},$$

where

$$\mathcal{D} = \left\{ y \in H : \begin{array}{c} y \text{ and } D_q y \text{ are } q\text{-regular at zero}, \\ y(d\pm) \text{ and } \mathcal{D}_{q^{-1}} y(d\pm) \text{ exist}, \\ y(0) - h_1 D_{q^{-1}} y(0) = 0, \\ y(a) + h_2 D_{q^{-1}} y(a) = 0, \\ y(d-) = \eta y(d+), \\ \mathcal{D}_{q^{-1}} y(d-) = \frac{1}{\eta} \mathcal{D}_{q^{-1}} y(d+), \text{ and} \\ \tau y \in H \end{array} \right\}.$$

Let $y, z \in \mathcal{D}$. Then we have

$$\int_0^a \left[(\tau y)(x) \overline{z(x)} - y(x) \overline{(\tau z)(x)} \right] d_q x$$
$$= [y,z](a) - [y,z](d+) + [y,z](d-) - [y,z](0), \tag{5}$$

where

$$[y,z] := y(\overline{D_{q^{-1}} z}) - (D_{q^{-1}} y) \overline{z}.$$

Theorem 1. *$\mathcal{T}$ is a positive self-adjoint operator in H.*

Proof. Let $y, z \in \mathcal{D}$. It follows from conditions (2)-(4) and (5) that

$$\langle \mathcal{T}y, z \rangle = \langle y, \mathcal{T}z \rangle. \tag{6}$$

Since $\mathcal{D}$ is a dense subset in H, we see that $\mathcal{T}$ *is a self-adjoint operator.*

Let

$$y(\zeta) = \begin{cases} y^{(1)}(\zeta), & \zeta \in (0,d) \\ y^{(2)}(\zeta), & \zeta \in (d,a), \end{cases} \quad y \in \mathcal{D}.$$

Then we get

$$\begin{aligned}
\langle \mathcal{T}y, y\rangle &= \int_0^d \left[-\frac{1}{q}D_{q^{-1}}D_q y^{(1)}(\zeta) + v(\zeta)\, y^{(1)}(\zeta)\right]\overline{y^{(1)}(\zeta)}d_q\zeta \\
&\quad + \int_d^a \left[-\frac{1}{q}D_{q^{-1}}D_q y^{(2)}(\zeta) + v(\zeta)\, y^{(2)}(\zeta)\right]\overline{y^{(2)}(\zeta)}d_q\zeta \\
&= \int_0^d -\frac{1}{q}\left(D_{q^{-1}}D_q y^{(1)}(\zeta)\right)\overline{y^{(1)}(\zeta)}d_q\zeta + \int_0^d v(\zeta)\left|y^{(1)}(\zeta)\right|^2 d_q\zeta \\
&\quad + \int_d^a -\frac{1}{q}\left(D_{q^{-1}}D_q y^{(2)}(\zeta)\right)\overline{y^{(2)}(\zeta)}d_q\zeta + \int_d^a v(\zeta)\left|y^{(2)}(\zeta)\right|^2 d_q\zeta \\
&= -\frac{1}{q}\int_0^d D_q\left[D_q y^{(1)}\left(q^{-1}\zeta\right)\right]\overline{y(\zeta)}d_q\zeta + \int_0^d v(\zeta)\left|y^{(1)}(\zeta)\right|^2 d_q\zeta \\
&\quad - \int_d^a D_q\left[D_q y^{(2)}\left(q^{-1}\zeta\right)\right]\overline{y^{(2)}(\zeta)}d_q\zeta + \int_d^a v(\zeta)\left|y^{(2)}(\zeta)\right|^2 d_q\zeta \\
&= -\frac{1}{q}\left[D_q y^{(1)}\left(q^{-1}d-\right)\overline{y^{(1)}(d-)}\right] + \frac{1}{q}\left[D_q y^{(1)}(0)\,\overline{y^{(1)}(0)}\right] \\
&\quad - \frac{1}{q}\left[D_q y^{(2)}\left(q^{-1}a\right)\overline{y^{(2)}(a)}\right] + \frac{1}{q}\left[D_q y^{(2)}(d+)\,\overline{y^{(2)}(d+)}\right] \\
&\quad + \frac{1}{q}\int_0^d \left|D_q y^{(1)}(\zeta)\right|^2 d_q\zeta + \int_0^d v(\zeta)\left|y^{(1)}(\zeta)\right|^2 d_q\zeta \\
&\quad + \frac{1}{q}\int_d^a \left|D_q y^{(2)}(\zeta)\right|^2 d_q\zeta + \int_d^a v(\zeta)\left|y^{(2)}(\zeta)\right|^2 d_q\zeta \\
&= \frac{1}{q}h_1\left|D_q y^{(1)}(0)\right|^2 + \frac{1}{q}h_2\left|D_q y^{(2)}\left(q^{-1}a\right)\right|^2 \\
&\quad + \frac{1}{q}\int_0^d \left|D_q y^{(1)}(\zeta)\right|^2 d_q\zeta + \int_0^d v(\zeta)\left|y^{(1)}(\zeta)\right|^2 d_q\zeta \\
&\quad + \frac{1}{q}\int_d^a \left|D_q y^{(2)}(\zeta)\right|^2 d_q\zeta + \int_d^a v(\zeta)\left|y^{(2)}(\zeta)\right|^2 d_q\zeta > 0,
\end{aligned}$$

due to $h_1, h_2 > 0$ and $v(\zeta) \geq 0$ for $\zeta \in [0, q^{-1}a]$. □

Let

$$u(\zeta) = \begin{cases} u^{(1)}(\zeta), & \zeta \in (0,d) \\ u^{(2)}(\zeta), & \zeta \in (d,a) \end{cases}$$

and

$$\chi(\zeta)=\begin{cases}\chi^{(1)}(\zeta), & \zeta\in(0,d)\\ \chi^{(2)}(\zeta), & \zeta\in(d,a)\end{cases}$$

be solutions of the problem

$$-\frac{1}{q}D_{q^{-1}}D_q y(\zeta)+v(\zeta)y(\zeta)=0,$$
$$y(d-)=\eta y(d+),$$
$$\mathcal{D}_{q^{-1}}y(d-)=\frac{1}{\eta}\mathcal{D}_{q^{-1}}y(d+),$$

satisfying

$$u^{(1)}(0)=h_1,\ D_{q^{-1}}u^{(1)}(0)=1,$$
$$\chi^{(2)}(a)=-h_2,\ D_{q^{-1}}\chi^{(2)}(a)=1.$$

Lemma 1. *Zero is not an eigenvalue of the operator $\mathcal{T}$.*

Proof. Let $y\in H$ and $\mathcal{T}y=0$. Then we have

$$-\frac{1}{q}D_{q^{-1}}D_q y(\zeta)+v(\zeta)y(\zeta)=0,$$

and

$$y(\zeta)=\begin{cases}c_1u^{(1)}(\zeta)+c_2\chi^{(1)}(\zeta), & \zeta\in(0,d)\\ c_3u^{(2)}(\zeta)+c_4\chi^{(2)}(\zeta), & \zeta\in(d,a),\end{cases}$$

where c_1,c_2,c_3 and c_4 are constants. From conditions (2)-(4), we infer that $y=0$. □

Definition 1. *The q-Wronskian of y and z is defined as*

$$W_q(y,z):=yD_qz-zD_qy.$$

Theorem 2. *Let*

$$G(\zeta,t)=-\frac{1}{W_q(u,\chi)}\begin{cases}u(\zeta)\chi(t), & 0\le\zeta\le t\le a,\ \zeta\neq d,\ t\neq d,\\ u(t)\chi(\zeta), & 0\le t\le\zeta\le a,\ \zeta\neq d,\ t\neq d.\end{cases}\tag{7}$$

Then $G(\zeta,t)$ is a q-Hilbert–Schmidt kernel, i.e.,

$$\int_0^d\int_0^d|G(\zeta,t)|^2d_q\zeta d_qt<\infty,\ \int_d^a\int_d^a|G(\zeta,t)|^2d_q\zeta d_qt<\infty.$$

Proof. By (7), we see that

$$\int_0^d d_q\zeta\int_0^d|G(\zeta,t)|^2d_qt<\infty,$$
$$\int_d^a d_q\zeta\int_d^a|G(\zeta,t)|^2d_qt<\infty$$

due to $u(.)\chi(.)\in H\times H$. Then, we obtain

$$\int_0^d\int_0^d|G(\zeta,t)|^2d_q\zeta d_qt<\infty,\ \int_d^a\int_d^a|G(\zeta,t)|^2d_q\zeta d_qt<\infty.\tag{8}$$

□

Theorem 3 ([11])**.** *Let A be an operator defined as*

$$A\{\zeta_i\} = \{y_i\},$$

where $i \in \mathbb{N} := \{1, 2, 3, ...\}$ and

$$y_i = \sum_{k=1}^{\infty} a_{ik}\zeta_k. \tag{9}$$

If

$$\sum_{i,k=1}^{\infty} |a_{ik}|^2 < \infty, \tag{10}$$

then A is compact in l^2.

Theorem 4. *Let $K : H \to H$ be an operator defined as*

$$(Kf)(\zeta) = \begin{cases} \int_0^d G(\zeta,\gamma) f^{(1)}(\gamma) d_q\gamma, & \zeta \in [0,d) \\ \int_d^a G(\zeta,\gamma) f^{(2)}(\gamma) d_q\gamma, & \zeta \in (d,a], \end{cases} \tag{11}$$

where

$$f(\zeta) = \begin{cases} f^{(1)}(\zeta), & \zeta \in [0,d) \\ f^{(2)}(\zeta), & \zeta \in (d,a], \end{cases} \quad f \in H.$$

Then K is a compact operator.

Proof. Let

$$\phi_i = \phi_i(\zeta) = \begin{cases} \phi_i^{(1)}(\zeta), & \zeta \in [0,d) \\ \phi_i^{(2)}(\zeta), & \zeta \in (d,a] \end{cases} \quad (i \in \mathbb{N})$$

be a complete, orthonormal basis of H. Let $i, k \in \mathbb{N}$. If we set

$$\zeta_i = \langle f, \phi_i \rangle = \int_0^d f^{(1)}(\zeta) \overline{\phi_i^{(1)}(\zeta)} d_q\zeta + \int_d^a f^{(2)}(\zeta) \overline{\phi_i^{(2)}(\zeta)} d_q\zeta,$$

$$y_i = \langle g, \phi_i \rangle = \int_0^d g^{(1)}(\zeta) \overline{\phi_i^{(1)}(\zeta)} d_q\zeta + \int_d^a g^{(2)}(\zeta) \overline{\phi_i^{(2)}(\zeta)} d_q\zeta,$$

$$a_{ik} = \int_0^d \int_0^d G(\zeta,t) \overline{\phi_i^{(1)}(\zeta)\phi_k^{(1)}(t)} d_q\zeta d_q t + \int_d^a \int_d^a G(\zeta,t) \overline{\phi_i^{(2)}(\zeta)\phi_k^{(2)}(t)} d_q\zeta d_q t.$$

Then, H is mapped isometrically l^2. Then, K transforms into A defined as (9) in l^2 and (8) is translated into (10). From Theorem 3, A is compact. Thus, K is compact. □

Since $K=\mathcal{T}^{-1}$, the completeness of the system of all eigenvectors of $\mathcal{T}$ is equivalent to the completeness of the system of all eigenvectors of K. By the Hilbert–Schmidt theorem, we obtain the following theorem.

Theorem 5. *For the boundary-value problem (1)-(4), there exists an orthonormal basis $\{\psi_k\}_{k\in\mathbb{N}}$ in H. For $f\in H$, we have*

$$f(\zeta)=\sum_{k=1}^{\infty}c_k\psi_k(\zeta), \tag{12}$$

where

$$c_k=\langle f,\psi_k\rangle,\ \ k\in\mathbb{N}.$$

Thus, we get

$$\lim_{N\to\infty}\left\{\begin{array}{c}\int_0^d\left|f^{(1)}(\zeta)-\sum_{k=1}^{N}c_k\psi_k^{(1)}(\zeta)\right|^2 d_q\zeta\\ +\int_d^a\left|f^{(2)}(\zeta)-\sum_{k=1}^{N}c_k\psi_k^{(2)}(\zeta)\right|^2 d_q\zeta\end{array}\right\}=0, \tag{13}$$

Moreover, it follows from (13) that

$$\int_0^d\left|f^{(1)}(\zeta)\right|^2 d_q\zeta+\int_d^a\left|f^{(2)}(\zeta)\right|^2 d_q\zeta=\sum_{k=1}^{\infty}|c_k|^2. \tag{14}$$

Now let's prove the main result of the article

Theorem 6. *Let $f,D_qf:[0,q^{-1}a]\to\mathbb{R}$ be continuous functions on $[0,d)\cup(d,q^{-1}a]$, has finite limits $f(d\pm)$, $D_{q^{-1}}f(d\pm)$ and satisfying conditions (2)-(4). Then the series*

$$f(\zeta)=\sum_{k=1}^{\infty}c_k\psi_k(\zeta), \tag{15}$$

where

$$c_k=\langle f,\psi_k\rangle,\ \ k\in\mathbb{N},$$

converges uniformly to f on the set $[0,d)\cup(d,a]$.

Proof. Let

$$\begin{aligned}S(y):=&\frac{1}{q}h_1\left|D_qy^{(1)}(0)\right|^2+\frac{1}{q}h_2\left|D_qy^{(2)}\left(q^{-1}a\right)\right|^2\\ &+\frac{1}{q}\int_0^d\left|D_qy^{(1)}(\zeta)\right|^2 d_q\zeta+\int_0^d v(\zeta)\left|y^{(1)}(\zeta)\right|^2 d_q\zeta\\ &+\frac{1}{q}\int_d^a\left|D_qy^{(2)}(\zeta)\right|^2 d_q\zeta+\int_d^a v(\zeta)\left|y^{(2)}(\zeta)\right|^2 d_q\zeta,\end{aligned} \tag{16}$$

and $S(y) \geq 0$. If we take

$$y = f(\zeta) - \sum_{k=1}^{N} c_k \psi_k(\zeta)$$

in (16), we conclude that

$$\begin{aligned}
&S\left(f(\zeta) - \sum_{k=1}^{N} c_k \psi_k(\zeta)\right) \\
&= \frac{1}{q} h_1 \left[D_q f^{(1)}(0) - \sum_{k=1}^{N} c_k D_q \psi_k^{(1)}(0)\right]^2 \\
&+ \frac{1}{q} h_2 \left[D_q f^{(2)}\left(q^{-1}a\right) - \sum_{k=1}^{N} c_k \left(D_q \psi_k^{(2)}(a)\right)\right]^2 \\
&+ \frac{1}{q} \int_0^d \left(D_q f^{(1)}(\zeta) - \sum_{k=1}^{N} c_k D_q \psi_k^{(1)}(\zeta)\right)^2 d_q\zeta \\
&+ \frac{1}{q} \int_d^a \left(D_q f^{(2)}(\zeta) - \sum_{k=1}^{N} c_k D_q \psi_k^{(2)}(\zeta)\right)^2 d_q\zeta \\
&+ \int_0^a v(\zeta) \left(f^{(1)}(\zeta) - \sum_{k=1}^{N} c_k \psi_k^{(1)}(\zeta)\right)^2 d_q\zeta \\
&+ \int_d^a v(\zeta) \left(f^{(2)}(\zeta) - \sum_{k=1}^{N} c_k \psi_k^{(2)}(\zeta)\right)^2 d_q\zeta \\
&= \frac{1}{q} h_1 \left[D_q f^{(1)}(0)\right]^2 + \frac{1}{q} h_2 \left[D_q f^{(2)}\left(q^{-1}a\right)\right]^2 \\
&- 2\frac{1}{q} \sum_{k=1}^{N} c_k \begin{bmatrix} -h_1 D_q f^{(1)}(0) D_q \psi_k^{(1)}(0) \\ -h_2 D_q f^{(2)}\left(q^{-1}a\right) D_q \psi_k^{(2)}(a) \end{bmatrix} \\
&- \frac{1}{q} \sum_{k,m=1}^{N} c_k c_m \begin{bmatrix} -h_1 D_q \psi_k^{(1)}(0) D_q \psi_m^{(1)}(0) \\ -h_2 D_q \psi_k^{(2)}(a) D_q \psi_m^{(2)}(a) \end{bmatrix} \\
&+ \frac{1}{q} \int_0^d (D_q f^{(1)}(\zeta))^2 d_q\zeta + \int_0^d v(\zeta) f^{(1)2}(\zeta) d_q\zeta \\
&+ \frac{1}{q} \int_d^a (D_q f^{(2)}(\zeta))^2 d_q\zeta + \int_d^a v(\zeta) f^{(2)2}(\zeta) d_q\zeta
\end{aligned}$$

$$-2\frac{1}{q}\sum_{k=1}^{N}c_k\begin{bmatrix}\int_0^d D_q f^{(1)}(\zeta)\, D_q\psi_k^{(1)}(\zeta)\, d_q\zeta \\ +\int_d^a D_q f^{(2)}(\zeta)\, D_q\psi_k^{(2)}(\zeta)\, d_q\zeta\end{bmatrix}$$
$$-2\sum_{k=1}^{N}c_k\begin{bmatrix}\int_0^d v(\zeta)\, f^{(1)}(\zeta)\,\psi_k^{(1)}(\zeta)\, d_q\zeta \\ +\int_d^a v(\zeta)\, f^{(2)}(\zeta)\,\psi_k^{(2)}(\zeta)\, d_q\zeta\end{bmatrix}$$
$$+\frac{1}{q}\sum_{k,m=1}^{N}c_kc_m\begin{bmatrix}\int_0^d D_q\psi_k^{(1)}(\zeta)\, D_q\psi_m^{(1)}(\zeta)\, d_q\zeta \\ +\int_d^a D_q\psi_k^{(2)}(\zeta)\, D_q\psi_m^{(2)}(\zeta)\, d_q\zeta\end{bmatrix}$$
$$+\sum_{k,m=1}^{N}c_kc_m\begin{bmatrix}\int_0^d v(\zeta)\,\psi_k^{(1)}(\zeta)\,\psi_m^{(1)}(\zeta)\, d_q\zeta \\ +\int_d^a v(\zeta)\,\psi_k^{(2)}(\zeta)\,\psi_m^{(2)}(\zeta)\, d_q\zeta\end{bmatrix}.$$

Applications of (2)-(4) and q-integration by parts give

$$\frac{1}{q}\int_0^d D_q\psi_k^{(1)}(\zeta)\, D_q f^{(1)}(\zeta)\, d_q\zeta+\int_0^d v(\zeta)\, f^{(1)}(\zeta)\,\psi_k^{(1)}(\zeta)\, d_q\zeta$$
$$+\frac{1}{q}\int_d^a D_q\psi_k^{(2)}(\zeta)\, D_q f^{(2)}(\zeta)\, d_q\zeta+\int_d^a v(\zeta)\, f^{(2)}(\zeta)\,\psi_k^{(2)}(\zeta)\, d_q\zeta$$
$$=\frac{1}{q}D_q\psi_k^{(1)}\left(q^{-1}d-\right) f^{(1)}(d-)-\frac{1}{q}D_q\psi_k^{(1)}(0)\, f^{(1)}(0)$$
$$+\frac{1}{q}D_q\psi_k^{(2)}\left(q^{-1}a\right) f^{(2)}(a)-\frac{1}{q}D_q\psi_k^{(2)}(d+)\, f^{(2)}(d+)$$
$$-\frac{1}{q}\int_0^d f^{(1)}(\zeta)\, D_{q^{-1}}\left(D_q\psi_k^{(1)}(\zeta)\right) d_q\zeta$$
$$-\frac{1}{q}\int_d^a f^{(2)}(\zeta)\, D_{q^{-1}}\left(D_q\psi_k^{(2)}(\zeta)\right) d_q\zeta$$
$$+\int_0^d v(\zeta)\, f^{(1)}(\zeta)\,\psi_k^{(1)}(\zeta)\, d_q\zeta+\int_d^a v(\zeta)\, f^{(2)}(\zeta)\,\psi_k^{(2)}(\zeta)\, d_q\zeta$$
$$=-\frac{1}{q}h_2 D_{q^{-1}}f^{(2)}(a)\, D_{q^{-1}}\psi_k^{(2)}(a)$$
$$-\frac{1}{q}h_1 D_{q^{-1}}f^{(1)}(0)\, D_q\psi_k^{(1)}(0)$$
$$+\int_0^d f^{(1)}(\zeta)\left[-\frac{1}{q}D_{q^{-1}}\left(D_q\psi_k^{(1)}(\zeta)\right)+v(\zeta)\,\psi_k^{(1)}(\zeta)\right] d_q\zeta$$
$$+\int_d^a f^{(2)}(\zeta)\left[-\frac{1}{q}D_{q^{-1}}\left(D_q\psi_k^{(2)}(\zeta)\right)+v(\zeta)\,\psi_k^{(2)}(\zeta)\right] d_q\zeta$$
$$=-h_2\frac{1}{q}D_{q^{-1}}f^{(2)}(a)\, D_{q^{-1}}\psi_k^{(2)}(a)$$
$$-h_1\frac{1}{q}D_{q^{-1}}f^{(1)}(0)\, D_q\psi_k^{(1)}(0)+\lambda_k c_k,$$

and

$$\frac{1}{q}\int_0^d D_q\psi_k^{(1)}(\zeta)\, D_q\psi_m^{(1)}(\zeta)\, d_q\zeta$$

$$+\frac{1}{q}\int_{d}^{a} D_q\psi_k^{(2)}(\zeta)\, D_q\psi_m^{(2)}(\zeta)\, d_q\zeta$$

$$+\int_{0}^{d} v(\zeta)\,\psi_k^{(1)}(\zeta)\,\psi_m^{(1)}(\zeta)\, d_q\zeta+\int_{d}^{a} v(\zeta)\,\psi_k^{(2)}(\zeta)\,\psi_m^{(2)}(\zeta)\, d_q\zeta$$

$$=\frac{1}{q}D_q\psi_m^{(1)}\left(q^{-1}d-\right)\psi_k^{(1)}(d-)+\frac{1}{q}D_q\psi_m^{(2)}\left(q^{-1}a\right)\psi_k^{(2)}(a)$$

$$-\frac{1}{q}D_q\psi_m^{(1)}(0)\,\psi_k^{(1)}(0)-\frac{1}{q}D_q\psi_m^{(2)}(d+)\,\psi_k^{(2)}(d+)$$

$$+\int_{0}^{d}\psi_k^{(1)}(\zeta)\left[-\frac{1}{q}D_{q^{-1}}\left(D_q\psi_m^{(1)}(\zeta)\right)+v(\zeta)\,\psi_m^{(1)}(\zeta)\right]d_q\zeta$$

$$+\int_{d}^{a}\psi_k^{(2)}(\zeta)\left[-\frac{1}{q}D_{q^{-1}}\left(D_q\psi_m^{(2)}(\zeta)\right)+v(\zeta)\,\psi_m^{(2)}(\zeta)\right]d_q\zeta$$

$$=\frac{1}{q}\psi_k^{(2)}(a)\, D_{q^{-1}}\psi_m^{(2)}(a)-\frac{1}{q}\psi_k^{(1)}(0)\, D_{q^{-1}}\psi_k^{(1)}(0)$$

$$+\lambda_k\left[\int_{0}^{d}\psi_k^{(1)}(\zeta)\,\psi_m^{(1)}(\zeta)\, d_q\zeta+\int_{d}^{a}\psi_k^{(2)}(\zeta)\,\psi_m^{(2)}(\zeta)\, d_q\zeta\right]$$

$$=-\frac{1}{q}h_1 D_{q^{-1}}\psi_k^{(2)}(a)\, D_{q^{-1}}\psi_m^{(2)}(a)$$

$$-\frac{1}{q}h_2 D_{q^{-1}}\psi_k^{(1)}(0)\, D_{q^{-1}}\psi_m^{(1)}(0)+\lambda_k\delta_{km},$$

where

$$\delta_{km}:=\begin{cases}1, & \text{if } k=m\\ 0, & \text{if } k\neq m.\end{cases}$$

Hence

$$S\left(f(\zeta)-\sum_{k=1}^{N}c_k\psi_k(\zeta)\right)=\frac{1}{q}h_1\left[D_qf^{(1)}(0)\right]^2$$

$$+\frac{1}{q}h_2\left[D_qf^{(2)}\left(q^{-1}a\right)\right]^2+\frac{1}{q}\int_{0}^{d}(D_qf^{(1)}(\zeta))^2 d_q\zeta$$

$$+\int_{0}^{d} v(\zeta)\, f^{(1)2}(\zeta)\, d_q\zeta+\frac{1}{q}\int_{d}^{a}(D_qf^{(2)}(\zeta))^2 d_q\zeta$$

$$+\int_{d}^{a} v(\zeta)\, f^{(2)2}(\zeta)\, d_q\zeta-\frac{1}{q}\sum_{k=1}^{N}\lambda_k c_k^2.$$

Then we get

$$\sum_{k=1}^{\infty}\lambda_k c_k^2\leq h_1\left[D_qf^{(1)}(0)\right]^2+h_2\left[D_qf^{(2)}\left(q^{-1}a\right)\right]^2$$

$$+\int_0^d (D_q f^{(1)}(\zeta))^2 d_q\zeta + q\int_0^d v(\zeta) f^{(1)2}(\zeta)\, d_q\zeta$$

$$+\int_d^a (D_q f^{(2)}(\zeta))^2 d_q\zeta + q\int_d^a v(\zeta) f^{(2)2}(\zeta)\, d_q\zeta. \tag{17}$$

due to S is nonnegative for all N. Thus, the convergence of the series

$$\sum_{k=1}^{\infty} \lambda_k c_k^2$$

follows.

Now, we shall prove that the series

$$\sum_{k=1}^{\infty} |c_k \psi_k(\zeta)| \tag{18}$$

is uniformly convergent on $[0,d)\cup(d,a]$. Since $\mathcal{T}\psi_k = \lambda_k\psi_k$, $k\in\mathbb{N}$, we have

$$\psi_k(\zeta) = \lambda_k(\mathcal{T}^{-1}\psi_k)(\zeta) = \lambda_k\langle G(\zeta,t), \psi_k\rangle,\ \ k\in\mathbb{N}.$$

If we rewrite the series (18), we conclude that

$$\sum_{k=1}^{\infty} |c_k\psi_k(\zeta)| = \sum_{k=1}^{\infty} \lambda_k |c_k \Upsilon_k(\zeta)|, \tag{19}$$

where

$$\Upsilon_k(\zeta) = \langle G(\zeta,t), \psi_k\rangle,\ \ k\in\mathbb{N}.$$

This can be regarded as the Fourier coefficients of $G(\zeta,t)$ as a function of t. From (17), we find

$$\sum_{k=1}^{\infty} \lambda_k \Upsilon_k^2(\zeta) \le h_1\left[D_q G^{(1)}(\zeta,0)\right]^2 + h_2\left[D_q G^{(2)}\left(\zeta, q^{-1}a\right)\right]^2$$

$$+\int_0^d (D_q G^{(1)}(\zeta,t))^2 d_q t + q\int_0^d v(t)\, G^{(1)2}(\zeta,t)\, d_q t$$

$$+\int_d^a (D_q G^{(2)}(\zeta,t))^2 d_q t + q\int_0^a v(t)\, G^{(2)2}(\zeta,t)\, d_q t.$$

Since all the functions appearing under the integral sign are bounded, we deduce that

$$\sum_{k=1}^{\infty} \lambda_k \Upsilon_k^2(\zeta) \le C,$$

where C is a constant. Applying the Cauchy–Schwartz inequality to the series (19), we see that

$$\sum_{k=n}^{n+m} \lambda_k \left| c_k \Upsilon_k(\zeta) \right| \le \sqrt{\sum_{k=n}^{n+m} \lambda_k c_k^2} \sqrt{\sum_{k=n}^{n+m} \lambda_k \Upsilon_k^2(\zeta)}$$

$$\le \sqrt{C} \sqrt{\sum_{k=n}^{n+m} \lambda_k c_k^2}. \tag{20}$$

From (17) and (20), the series (18) is uniformly convergent on $[0, d) \cup (d, a]$. Since

$$\left| \sum_{k=1}^{\infty} c_k \psi_k(\zeta) \right| \le \sum_{k=1}^{\infty} \left| c_k \psi_k(\zeta) \right|,$$

the series (15) is also uniformly convergent on $[0, d) \cup (d, a]$.

Let

$$f_1(\zeta) = \sum_{k=1}^{\infty} c_k \psi_k(\zeta). \tag{21}$$

Since the series (21) is uniformly convergent on $[0, d) \cup (d, a]$, we get

$$\int_0^d f_1^{(1)}(\zeta)\, \psi_k^{(1)}(\zeta)\, d_q\zeta + \int_d^a f_1^{(2)}(\zeta)\, \psi_k^{(2)}(\zeta)\, d_q\zeta = c_k \ (k \in \mathbb{N}).$$

Consequently, the Fourier coefficients of f and f_1 are the same. Applying the Parseval equality (14) to the function $f - f_1$, we find $f - f_1 = 0$, due to the Fourier coefficients of the function $f - f_1$ are zero. This finishes the proof. □

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