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TOTAL COALITION GRAPHS OF CYCLES AND PATHS

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Abstract: A subset of vertices in a graph G is a total dominating set if every vertex of G is adjacent to at least one vertex within the subset. Two non-total dominating sets form a total coalition in a graph if their union is a total dominating set. A partition π of graph vertices into non-total dominating sets is a total coalition partition if every set of π forms a total coalition set with at least one other set of π. Vertices of the total coalition graph $TCG(G,\pi)$ correspond with the sets of π, and two vertices are adjacent in $TCG(G,\pi)$ if and only if the corresponding sets constitute a total coalition. We show that C_{4k} is a universal total coalition cycle for $k \geq 2$, that is, a cycle whose total coalition partitions generate all possible total coalition graphs of cycles. We also demonstrate that P_n is a universal total coalition path for $n \geq 5$.

Keywords: total coalition graph, total dominating set.

1 Introduction

We consider finite, undirected and simple graphs with no isolated vertices. The vertex set of a graph G is denoted by $V(G)$. We generally follow the notation and graph theory terminology from the book [22]. A dominating

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set in a graph G is a subset $S \subseteq V(G)$, such that every vertex outside S is adjacent to at least one vertex in S. The domination in graphs appears as a model for facility location problems, and it has found many applications, in particular, in design and analysis of transportation and wireless sensor networks [18]. Detailed information on graph domination and related subjects can be found in books [16, 17, 18, 19]. Cockayne, Dawes, and Hedetniemi proposed a variation of domination known as the total domination [8]. A subset $D \subseteq V(G)$ is a total dominating set of a graph G if every vertex in G is adjacent to at least one vertex in D. For a comprehensive monograph on total dominating sets, we refer the reader to [21]. In 2020, Haynes et al. presented a novel graph invariant, known as the coalition, based on dominating sets in graphs [11]. Forming coalitions in the industrial sector can be used to address challenges or attain a shared objective [7].

In a graph G, a coalition consists of two disjoint subsets $V_1, V_2 \subset V(G)$, such that neither V_1 nor V_2 is a dominating set, but the union $V_1 \cup V_2$ is a dominating set. A coalition partition of $V(G)$ is a vertex partition $\pi(G) = \{V_1, V_2, \ldots, V_k\}$ where each set V_i is either a single dominating vertex or forms a coalition with another set V_j for every $i \in \{1, 2, \ldots, k\}$. Haynes et al. initiated the study of coalitions graphs in [14]. To describe the formation of coalitions in $\pi(G)$, they associate with the partition its coalition graph $CG(G, \pi)$. Vertices of this graphs correspond to the sets of the partition, and two vertices are adjacent if and only if the corresponding sets form a coalition. A path is called coalition universal if its coalition partitions define all possible coalition graphs of paths. In [14], the authors demonstrated that there are only 18 coalition graphs of paths. Henning et al. proved that there are no universal coalition paths and P_{10} is the shortest path that defines the maximal number of coalition graphs [5]. Haynes et al. showed that there are precisely 27 graphs of order at most 6 that can be coalition graphs of cycles [14]. They asked about the shortest cycle having the maximum number of coalition graphs. Dobrynin and Golmohammadi showed that C_{15} is the shortest cycle satisfying this property [10].

A significant variation of the coalition concept is the total coalition [1]. A total coalition in a graph G consists of two disjoint subsets of vertices V_1 and V_2, neither of which is a total dominating set, but whose union is a total dominating set. A total coalition partition $\pi(G) = \{V_1, V_2, \ldots, V_k\}$ is a partition of vertices of G into non-total dominating sets, such that each set of π forms a total coalition with another set of π. The maximum cardinality of a total coalition partition is called the total coalition number of a graph G and denoted by $TC(G)$. The total coalition graph $TCG(G, \pi)$, defined by a total coalition partition π, is constructed by the same way as the coalition graph $CG(G, \pi)$. This concept has been recently introduced by Barát and Blázsik in [6]. To get some insights into results on the coalition and its variations, we refer to the articles [2, 3, 4, 9, 12, 13, 15, 20].

In this paper, we show that C_{4k} is the universal total coalition cycle for $k \geq 2$ and P_n is the total coalition universal path for $n \geq 5$.

2 Main results

In this section, the total coalition graphs of paths and cycles will be described. We start with the following definitions.

Definition 1. *A path is called a universal path if its total coalition partitions define all possible total coalition graphs of paths.*

Definition 2. *A cycle is called a universal cycle if its total coalition partitions define all possible total coalition graphs of cycles.*

According to the following known result, we realize that the number of total coalition graphs of cycles is finite.

Theorem 1. [1] *For any cycle C_n,*

$$TC(C_n) = \begin{cases} 4, & n \equiv 0 \ (mod\ 4) \\ 3, & \text{otherwise.} \end{cases}$$

The next result gives an upper bound for the maximum vertex degree Δ of a total coalition graph.

Lemma 1. [6] *The maximum vertex degree of $TCG(G, \pi)$ cannot be greater than the maximum vertex degree of G, i.e., $\Delta(TCG(G, \pi)) \leq \Delta(G)$.*

By Lemma 1, for a vertex v of a total coalition graph of cycles, $\deg(v) \leq 2$. Theorem 1 shows that the order of a total coalition graph of C_n is at most 4 for $n \geq 4$. Then the possible total coalition graphs of C_n are K_2, P_3, K_3, $2K_2$, P_3, P_4, and C_4. It is easy to see that C_3 has the unique total coalition partition $\pi = \{\{v_1\}, \{v_2\}, \{v_3\}\}$ and $TCG(C_3, \pi) \cong C_3$. Further assume that C_n is a cycle of order $n \geq 4$. We have calculated the number of total coalition graphs for cycles of small order. These results are collected in Table 1.

TABLE 1. Number of total coalition graphs of C_n.

$TCG(C_n)$	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}	C_{18}
P_3	2	10	42	112	338	882	2350	6072	15638	39130	97762	243040	601218	1476450	3617502
K_2	1	5	16	35	81	180	391	825	1726	3575	7351	15020	30561	61965	125296
K_3	.	5	6	21	24	85	150	341	600	1365	2646	5461	10584	21845	43350
$2K_2$	.	.	.	.	64	.	.	.	1530	.	.	.	28864	.	.
P_4	.	.	.	.	32	.	.	.	390	.	.	.	3392	.	.
C_4	1	.	.	.	1	.	.	.	1	.	.	.	1	.	.

Proposition 1. *The cycle C_n defines the total coalition graphs K_2, P_3, and K_3 for $n \geq 5$.*

Proof. Let $V(C_n) = (v_1, v_2, \ldots, v_n)$. In order to prove the proposition, we present three total coalition partitions π_1, π_2, π_3 of $V(C_n)$ that generate the total coalition graphs K_2, P_3, K_3, respectively. Let $\pi_1 = \{V_1, V_2\}$ and $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2, v_4, v_5, \ldots, v_n\}$. It can be seen that none of two sets of π_1 is total dominating, and together these sets form a total coalition. Hence, $TCG(C_n, \pi_1) \cong K_2$.

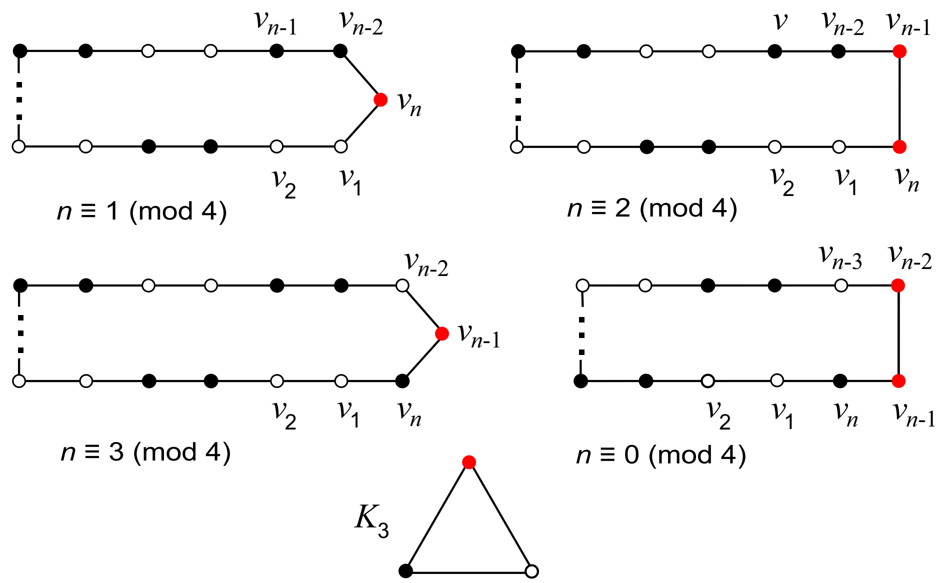

FIG. 1. Total coalition partitions of C_n for K_3.

Next assume that $\pi_2 = \{V_1, V_2, V_3\}$ with $V_1 = \{v_1\}$, $V_2 = \{v_3\}$, and $V_3 = \{v_2, v_4, v_5, \ldots, v_n\}$. The sets V_1 and V_3 form a total coalition, as do the sets V_2 and V_3, while the union $V_1 \cup V_2$ is not a total domination set. Hence, $TCG(C_n, \pi_2) \cong P_3$.

For $\pi_3 = \{V_1, V_2, V_3\}$, we consider four cases. Total coalition partitions for these cases are depicted in Fig. 1.

Case 1. Let $n \equiv 1 \pmod 4$. A suitable total coalition partition of C_n consists of the following sets: $V_1 = \cup_{i=1}^{(n-1)/4}\{v_{4i-3}, v_{4i-2}\}$ (white vertices), $V_2 = \cup_{i=1}^{(n-1)/4}\{v_{4i-1}, v_{4i}\}$ (black vertices), and $V_3 = \{v_n\}$.

Case 2. Let $n \equiv 2 \pmod 4$. In this case, $V_1 = \cup_{i=1}^{(n-2)/4}\{v_{4i-3}, v_{4i-2}\}$, $V_2 = \cup_{i=1}^{(n-2)/4}\{v_{4i-1}, v_{4i}\}$, and $V_3 = \{v_{n-1}, v_n\}$.

Case 3. Let $n \equiv 3 \pmod 4$. Then $V_1 = \cup_{i=1}^{(n-3)/4}\{v_{4i-3}, v_{4i-2}\} \cup \{v_{n-2}\}$, $V_2 = \cup_{i=1}^{(n-3)/4}\{v_{4i-1}, v_{4i}\} \cup \{v_n\}$, and $V_3 = \{v_{n-1}\}$.

Case 4. Let $n \equiv 0 \pmod 4$. We can take the following partition of vertices: $V_1 = \cup_{i=1}^{(n-4)/4}\{v_{4i-3}, v_{4i-2}\} \cup \{v_{n-3}\}$, $V_2 = \cup_{i=1}^{(n-4)/4}\{v_{4i-1}, v_{4i}\} \cup \{v_n\}$, and $V_3 = \{v_{n-2}, v_{n-1}\}$.

It not hard to verify that none of the sets of π_3 is a total dominating, but each pair of the sets forms a total coalition. Then $TCG(C_n, \pi_3) \cong K_3$. □

Proposition 2. *The cycle C_{4k} defines the total coalition graphs C_4, $2K_2$, and P_4 for $k \geq 2$.*

Proof. Let $V(C_n) = (v_1, v_2, \ldots, v_n)$. We first present a total coalition partition π for C_{4k} whose the total coalition graph is C_4. Let $\pi = \{V_1, V_2, V_3, V_4\}$,

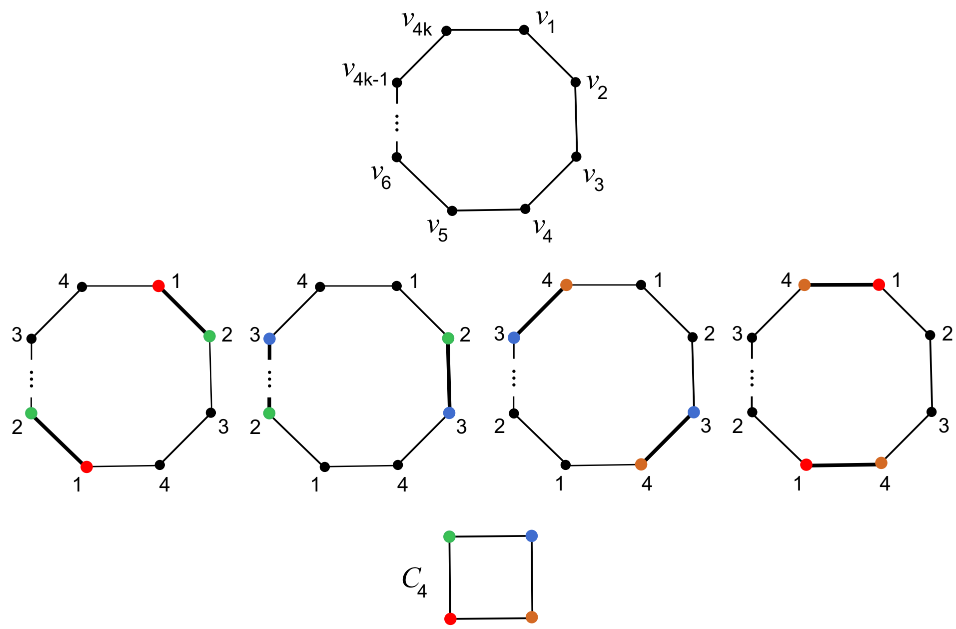

FIG. 2. A total coalition partition of C_{4k} for C_4.

such that $V_1 = \cup_{i=1}^{n/4}\{v_{4i-3}\}$, $V_2 = \cup_{i=1}^{n/4}\{v_{4i-2}\}$, $V_3 = \cup_{i=1}^{n/4}\{v_{4i-1}\}$, and $V_4 = \cup_{i=1}^{n/4}\{v_{4i}\}$. For an illustration, the vertex numbering of C_{4k} and indices of the corresponding partition sets of π are shown in Fig. 2. It is easy to see that only pairs $\{V_1, V_4\}$ and $\{V_i, V_{i+1}\}$ for $i = 1, 2, 3$ form total coalitions. Hence, $TCG(C_{4k}, \pi) \cong C_4$.

We now proceed to prove that C_{4k} generates the total coalition graph $2K_2$. The partition $\pi_1 = \{V_1 = \{v_1, v_2, v_5\}, V_2 = \{v_3, v_4, v_7\}, V_3 = \{v_6\}, V_4 = \{v_8\}\}$ is a total coalition partition of C_8 in which only two pairs $\{V_1, V_3\}$ and $\{V_2, V_4\}$ form total coalitions (see Fig. 3). Indeed, $V_2 \cup V_3$ and $V_3 \cup V_4$ are not dominating sets, while $V_1 \cup V_2$ and $V_1 \cup V_4$ are not total dominating sets. Then $TCG(C_8, \pi_1) \cong 2K_2$.

Now we construct a total coalition partition of the cycle C_{12} by adding four new vertices between two vertices of C_8 labeled 1 and 2 as shown in Fig. 3. A total coalition partition π_2 for C_{12} is constructed from π_1 by adding one vertex to every set of π_1. Then the pairs of sets that form total coalitions in π_2 are the same as for π_1. Consequently, $V_1 \cup V_3$ and $V_2 \cup V_4$ are the total dominating sets of C_{12}. This implies $TCG(C_{12}, \pi_2) \cong 2K_2$. Analogously, we get the total coalition partition π_3 of the cycle C_{16} with $TCG(C_{16}, \pi_3) \cong 2K_2$ by inserting four new vertices into C_{12}. If we continue in this manner, we conclude that $TCG(C_{4k}, \pi_{k-1}) \cong 2K_2$.

Finally we show that C_{4k} defines the total coalition graph P_4 by applying the approach from the previous case. Let $\pi_1 = \{V_1 = \{v_1, v_2, v_5\}$, $V_2 = \{v_3\}, V_3 = \{v_4, v_7, v_8\}, V_4 = \{v_6\}\}$ be a partition of C_8 (see Fig. 4). It is clear

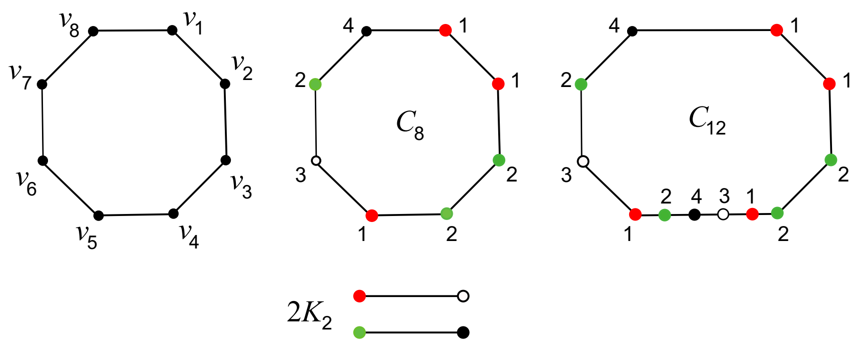

FIG. 3. Total coalition partitions of C_8 and C_{12} for $2K_2$.

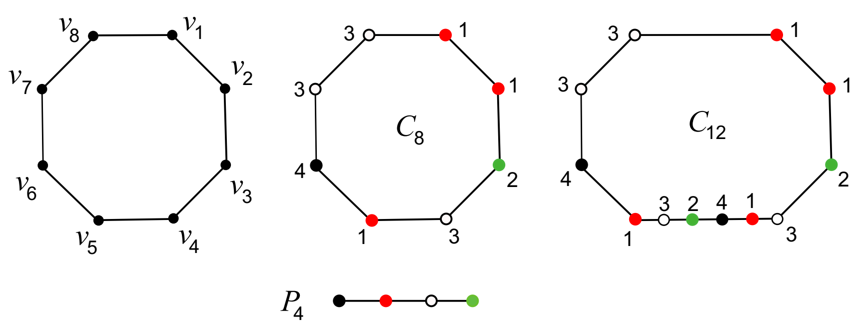

FIG. 4. Total coalition partitions of C_8 and C_{12} for P_4.

that pairs $\{V_1, V_3\}$, $\{V_1, V_4\}$, and $\{V_2, V_3\}$ form total coalitions, while $V_1 \cup V_2$, $V_2 \cup V_4$, and $V_3 \cup V_4$ are not dominating sets. Then $TCG(C_8, \pi_1) \cong P_4$.

Now we construct a total coalition partition of the cycle C_{12} by adding four new vertices between two vertices of C_8 labeled 1 and 3 as illustrated in Fig. 4. A total coalition partition π_2 of C_{12} is obtained from π_1 by adding one vertex to every set of π_1. Then the same pairs of sets form total coalitions in π_1 and π_2. Therefore, $TCG(C_{12}, \pi_2) \cong P_4$. By continuing this pattern, we infer that $TCG(C_{4k}, \pi_{k-1}) \cong P_4$. □

Propositions 1 and 2 lead to the following corollary.

Corollary 1. *The cycle C_{4k} is the universal total coalition cycle for $k \geq 2$.*

Now we turn our attention to total coalition graphs of paths. Their total coalition numbers were determined in [1].

Proposition 3. [1] *For any path P_n of order $n \geq 3$,*

$$TC(P_n) = \begin{cases} 2, & \text{if } n = 4 \\ 3, & \text{otherwise.} \end{cases}$$

By Lemma 1 and Proposition 3, the possible total coalition graphs of P_n are K_2 and P_3 for all $n \geq 3$. The number of total coalition graphs for the paths of small order is presented in Table 2.

Table 2. Number of total coalition graphs of P_n.

$TCG(P_n)$	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}
K_2	1	4	11	23	48	103	217	448	919	1879	3824	7751	15669	31612	63667	128047
P_3	1	.	3	12	30	84	239	620	1564	3976	10033	24948	61622	151844	372851	912084

Proposition 4. *The path P_n defines the total coalition graphs K_2 and P_3 for $n \geq 5$.*

Proof. Let $V(P_n) = (v_1, v_2, \dots, v_n)$. To prove the proposition, we provide two total coalition partitions π_1 and π_2 of P_n that generate the total coalition graphs K_2 and P_3, respectively. Let $\pi_1 = \{V_1, V_2\}$ and $V_1 = \{v_1, v_2\}$, $V_2 = \{v_3, v_4, \dots, v_n\}$. We observe that the sets V_1 and V_2 form a total coalition. Hence, $TCG(P_n, \pi_1) \cong K_2$.

Next consider partition $\pi_2 = \{V_1, V_2, V_3\}$, such that $V_1 = \{v_1\}$, $V_2 = \{v_3\}$, and $V_3 = \{v_2, v_4, \dots, v_n\}$. The set V_3 forms a total coalition with each set of the partition π_2, while the union $V_1 \cup V_3$ is not a domination set. Therefore, $TCG(P_n, \pi_2) \cong P_3$. $\square$

As a consequence of Proposition 4, we get the following result.

Corollary 2. *The path P_n is the universal total coalition path for $n \geq 5$.*

In conclusion, we state the following open problem.

Problem 1. *Characterize the total coalition graphs of trees.*

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