NORMAL OPERATORS FOR MOMENTUM RAY TRANSFORMS, II: SAINT-VENANT OPERATORS

SHUBHAM R. JATHAR [iD], MANAS KAR [iD], VENKATESWARAN P. KRISHNAN [iD], AND VLADIMIR A. SHARAFUTDINOV [iD]

Communicated by E.M. Rudoy

Abstract: The momentum ray transform I_m^k integrates a rank m symmetric tensor field f on $\mathbb{R}^n$ over lines with the weight t^k, $I_m^k f(x,\xi) = \int_{-\infty}^{\infty} t^k \langle f(x+t\xi), \xi^m \rangle \, \mathrm{d}t$. Let $N_m^k = (I_m^k)^* I_m^k$ be the normal operator of I_m^k. To what extent is a symmetric m-tensor field f determined by the data $(N_m^0 f, \ldots, N_m^r f)$ for some $0 \le r \le m$? The Saint Venant operator W_m^r is a linear differential operator of order $m-r$ with constant coefficients on the space of symmetric m-tensor fields. We derive an explicit formula expressing $W_m^r f$ in terms of $(N_m^0 f, \ldots, N_m^r f)$. The tensor field $W_m^r f$ represents the

full local information on f that can be extracted from the data $(N_m^0 f, \ldots, N_m^r f)$.

Keywords: ray transform, inverse problems, Saint-Venant operator, tensor tomography, momentum ray transform.

1 Introduction

This article is a follow-up to our prior work [4]. To ensure a self-contained presentation, we have chosen to provide only a condensed version in the introduction and Section 2. We refer the reader to [4] for more details.

Let f be a Schwartz class symmetric m-tensor field on $\mathbb{R}^n$. The kth momentum ray transform $I_m^k f$ of f is defined by

$$I_m^k f(x, \xi) = \int_{\mathbb{R}} t^k f_{i_1 \cdots i_m}(x + t\xi)\xi^{i_1} \cdots \xi^{i_m} \, \mathrm{d}t \tag{1}$$

$$\left(x \in \mathbb{R}^n, \ \xi \in \mathbb{R}^n, \ |\xi| = 1, \ \langle x, \xi \rangle = 0\right).$$

As in (1), with repeating indices, the Einstein summation convention is used throughout the article.

Momentum ray transforms are used as the main tool in the study of higher order versions of the Calderón inverse problem, see [1, 5, 2].

Let $(I_m^k)^*$ be the L^2-adjoint of I_m^k. Instead of working directly with the momentum ray transforms, we work with the associated normal operators $N_m^k = (I_m^k)^* I_m^k$. Being an averaging operator, N_m^k represents a better measurement model than the momentum ray transforms themselves. An inversion formula was obtained in [4] which recovers a symmetric m-tensor f from the data $(N_m^0 f, \ldots, N_m^m f)$; the formula is reproduced in Theorem 1 below.

In this work we investigate the problem of recovering a tensor field from partial data. To what extent is a symmetric m-tensor field f determined by the data $(N_m^0 f, \ldots, N_m^r f)$ for some $0 \leq r \leq m$?

In the next section, we recall the definition of the Saint Venant operator

$$W_m^r : C^\infty(\mathbb{R}^n; S^m) \to C^\infty(\mathbb{R}^n; S^{m-r} \otimes S^m) \quad (0 \leq r \leq m). \tag{2}$$

It is a linear differential operator of order $m - r$ with constant coefficients. This operator was briefly mentioned in [8, Theorem 2.17.2], but the operator $W = W_m^0$ was widely used throughout Chapter 2 of [8]. It is closely related to the equation

$$dv = f, \tag{3}$$

where $d = \sigma\nabla$ is the inner derivative defined in Section 2.3 below. Namely, the equation (3) is solvable in a simply connected domain $U \subset \mathbb{R}^n$ if and only if the right-hand side satisfies $W_m^0 f = 0$, see [8, Theorem 2.2.2]. Quite similarly, $W_m^r f = 0$ is the consistency condition for the equation $d^{r+1} v = f$, see [8, Theorem 2.17.2]. In the case $m = 2$, the condition $W_2^0 f = 0$ is popular in linear elasticity and is called the *deformation consistency condition*; it was obtained by Saint-Venant.

For $f \in \mathcal{S}(\mathbb{R}^n; S^m)$, the tensor field $W_m^r f$ represents the full *local* information on f that can be extracted from the data $(I_m^0 f, \ldots, I_m^r f)$, see [8, Theorem 2.17.2]. In particular, $W_m^r f$ is uniquely determined by $(N_m^0 f, \ldots, N_m^r f)$. The paper [7] establishes that for $f \in \mathcal{S}(S^m)$ and $0 \leq r \leq m$, the tensor field $W_m^r f$ can be explicitly recovered from $(I_m^0 f, \ldots, I_m^r f)$. In [6, Theorem 3.1], the kernel of the momentum ray transform is described using the Saint Venant operator. It is shown that for $f \in \mathcal{S}(S^m)$, $(I_m^0 f, \ldots, I_m^r f) = 0$ if and only if $W_m^r f = 0$. We will derive an explicit formula expressing $W_m^r f$ through $(N_m^0 f, \ldots, N_m^r f)$; see Theorem 2 below.

2   Basic definitions and main result

2.1. Tensor algebra. Let $T\mathbb{R}^n = \oplus_{m=0}^\infty T^m \mathbb{R}^n$ be the complex tensor algebra over $\mathbb{R}^n$. Assuming n to be fixed, the notation $T^m \mathbb{R}^n$ will be often abbreviated to T^m. For a fixed orthonormal basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$, by $u_{i_1 \ldots i_m} = u^{i_1 \ldots i_m} = u(e_{i_1}, \ldots, e_{i_m})$ we denote *coordinates* (= *components*) of a tensor $u \in T^m$ with respect to the basis. There is no distinction between covariant and contravariant tensors since we use orthonormal bases only. The standard dot product on $\mathbb{R}^n$ extends to T^m by

$$\langle u, v \rangle = u^{i_1 \ldots i_m} \overline{v_{i_1 \ldots i_m}}.$$

Let $S^m = S^m \mathbb{R}^n$ be the subspace of T^m consisting of symmetric tensors. *The partial symmetrization* $\sigma(i_1 \ldots i_m) : T^{m+k} \to T^{m+k}$ *in the indices* $(i_1, \ldots, i_m)$ is defined by

$$\sigma(i_1 \ldots i_m) u_{i_1 \ldots i_m j_1 \ldots j_k} = \frac{1}{m!} \sum_{\pi \in \Pi_m} u_{i_{\pi(1)}, \ldots, i_{\pi(m)} j_1 \ldots j_k},$$

where the summation is performed over the group Π_m of all permutations of the set $\{1, \ldots, m\}$. In particular, $\sigma : T^m \to S^m$ is the symmetrization in all indices. Given $u \in S^m$ and $v \in S^k$, *the symmetric product* $uv \in S^{m+k}$ is defined by $uv = \sigma(u \otimes v)$. Being equipped with the symmetric product, $S^* \mathbb{R}^n = \bigoplus_{m=0}^\infty S^m \mathbb{R}^n$ becomes a commutative graded algebra that is called *the algebra of symmetric tensors over* $\mathbb{R}^n$.

Given $u \in S^m$, let $i_u : S^k \to S^{m+k}$ be the operator of symmetric multiplication by u and let $j_u : S^{m+k} \to S^k$ be the adjoint of i_u. These operators are written in coordinates as

$$(i_u v)_{i_1 \ldots i_{m+k}} = \sigma(i_1 \ldots i_{m+k}) u_{i_1 \ldots i_m} v_{i_{m+1} \ldots i_{m+k}}$$

$$(j_u v)_{i_1 \ldots i_k} = v_{i_1 \ldots i_{m+k}} u^{i_{k+1} \ldots i_{m+k}}.$$

For the Kronecker tensor δ, the notations i_δ and j_δ will be abbreviated to i and j respectively.

2.2. Tensor fields. Recall that the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the topological vector space consisting of C^∞-smooth complex-valued functions on $\mathbb{R}^n$ that decay rapidly at infinity together with all derivatives, equipped with the standard topology. Let $\mathcal{S}(\mathbb{R}^n; S^m) = \mathcal{S}(\mathbb{R}^n) \otimes S^m$ be the topological

vector space of smooth fast decaying symmetric m-tensor fields, defined on $\mathbb{R}^n$. In Cartesian coordinates, such a tensor field is written as $f = (f_{i_1...i_m})$ with coordinates (= components) $f_{i_1...i_m} = f^{i_1...i_m} \in \mathcal{S}(\mathbb{R}^n)$ symmetric in all indices.

We use *the Fourier transform* $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, $f \mapsto \widehat{f}$ in the form (hereafter i is the imaginary unit)

$$\mathcal{F}f(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle y,x \rangle} f(x) \, \mathrm{d}x.$$

The Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n; S^m) \to \mathcal{S}(\mathbb{R}^n; S^m)$, $f \mapsto \widehat{f}$ of symmetric tensor fields is defined component-wise:

$$\widehat{f}_{i_1...i_m} = \widehat{f_{i_1...i_m}}.$$

The L^2-product on $C_0^\infty(\mathbb{R}^n; T^m)$ is defined by

$$(f,g)_{L^2(\mathbb{R}^n;T^m)} = \int_{\mathbb{R}^n} \langle f(x), g(x) \rangle \, dx. \tag{4}$$

2.3. Inner derivative and divergence. The first-order differential operator

$$d : C^\infty(\mathbb{R}^n; S^m) \to C^\infty(\mathbb{R}^n; S^{m+1})$$

defined by

$$(df)_{i_1...i_{m+1}} = \sigma(i_1 \ldots i_{m+1}) \frac{\partial f_{i_1...i_m}}{\partial x^{i_{m+1}}} = \frac{1}{m+1} \left(\frac{\partial f_{i_2...i_{m+1}}}{\partial x^{i_1}} + \cdots + \frac{\partial f_{i_1...i_m}}{\partial x^{i_{m+1}}} \right)$$

is called *the inner derivative*.

The divergence

$$\mathrm{div} : C^\infty(\mathbb{R}^n; S^{m+1}) \to C^\infty(\mathbb{R}^n; S^m)$$

is defined by

$$(\mathrm{div}\, f)_{i_1...i_m} = \delta^{jk} \frac{\partial f_{i_1...i_m j}}{\partial x^k}.$$

The operators d and $-\mathrm{div}$ are formally adjoint to each other with respect to the L^2-product (4).

2.4. The space $\mathcal{S}(T\mathbb{S}^{n-1})$. The Schwartz space $\mathcal{S}(E)$ is well-defined for a smooth vector bundle $E \to M$ over a compact manifold with the help of a finite atlas and partition of unity subordinate to the atlas.

In particular, the Schwartz space $\mathcal{S}(T\mathbb{S}^{n-1})$ is well defined for the tangent bundle

$$T\mathbb{S}^{n-1} = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{S}^{n-1} : \langle x,\xi \rangle = 0\} \to \mathbb{S}^{n-1}, \quad (x,\xi) \mapsto \xi$$

of the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

The Fourier transform $\mathcal{F} : \mathcal{S}(T\mathbb{S}^{n-1}) \to \mathcal{S}(T\mathbb{S}^{n-1})$, $\varphi \mapsto \widehat{\varphi}$ is defined by

$$\mathcal{F}\varphi(y,\xi) = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\xi^\perp} e^{-i\langle y,x \rangle} \varphi(x,\xi) \, \mathrm{d}x,$$

where dx is the $(n-1)$-dimensional Lebesgue measure on the hyperplane $\xi^\perp = \{x \in \mathbb{R}^n : \langle \xi, x \rangle = 0\}$.

The L^2-product on $\mathcal{S}(T\mathbb{S}^{n-1})$ is defined by

$$(\varphi, \psi)_{L^2(T\mathbb{S}^{n-1})} = \int\limits_{\mathbb{S}^{n-1}} \int\limits_{\xi^\perp} \varphi(x,\xi)\overline{\psi(x,\xi)} \, dx \, d\xi, \tag{5}$$

where $d\xi$ is the $(n-1)$-dimensional Euclidean volume form on the unit sphere $\mathbb{S}^{n-1}$.

2.5. Momentum ray transform.

It is convenient to parameterize the family of oriented lines in $\mathbb{R}^n$ by points of the manifold $T\mathbb{S}^{n-1}$. Namely, a point $(x,\xi) \in T\mathbb{S}^{n-1}$ determines the line $\{x + t\xi : t \in \mathbb{R}\}$ through x in the direction ξ.

For an integer $k \geq 0$, *the momentum ray transform*

$$I_m^k : \mathcal{S}(\mathbb{R}^n; S^m) \to \mathcal{S}\left(T\mathbb{S}^{n-1}\right)$$

is the linear continuous operator defined by (1).

2.6. Normal operators.

The formal adjoint of the momentum ray transform I_m^k with respect to L^2-products (4) and (5)

$$\left(I_m^k\right)^* : \mathcal{S}\left(T\mathbb{S}^{n-1}\right) \to C^\infty\left(\mathbb{R}^n; S^m\right)$$

is expressed by

$$\left((I_m^k)^*\varphi\right)_{i_1\ldots i_m}(x) = \int_{\mathbb{S}^{n-1}} \langle x,\xi\rangle^k \xi_{i_1} \ldots \xi_{i_m} \varphi\big(x - \langle x,\xi\rangle\xi, \xi\big) \, d\xi.$$

We emphasize that, for $\varphi \in \mathcal{S}(T\mathbb{S}^{n-1})$, the tensor field $(I_m^k)^*\varphi$ does not need to fast decay at infinity.

Let

$$N_m^k = (I_m^k)^* I_m^k : \mathcal{S}\left(\mathbb{R}^n; S^m\right) \to C^\infty\left(\mathbb{R}^n; S^m\right)$$

be the normal operator for the momentum ray transform I_m^k. For $f \in \mathcal{S}\left(\mathbb{R}^n; S^m\right)$, the Fourier transform $\widehat{N_m^k f} \in \mathcal{S}'\left(\mathbb{R}^n; S^m\right)$ is well defined at least in the distribution sense and the restriction of $\widehat{N_m^k f}$ to $\mathbb{R}^n \setminus \{0\}$ belongs to $C^\infty\left(\mathbb{R}^n \setminus \{0\}; S^m\right)$.

2.7. The inversion formula.

Let Γ be Euler's Gamma function and let the operator $(-\Delta)^{1/2}$ be defined with the help of the Fourier transform by $|y|\mathcal{F} = \mathcal{F}(-\Delta)^{1/2}$. We use the definition

$$(2l+1)!! = 1 \cdot 3 \cdots (2l+1), \quad (-1)!! = 1.$$

Let us reproduce [4, Theorem 3.1].

Theorem 1. *Given integers $m \geq 0$ and $n \geq 2$, a tensor field $f \in \mathcal{S}(\mathbb{R}^n; S^m)$ is recovered from the data $(N_m^0 f, N_m^1 f, \ldots, N_m^m f)$ by the inversion formula*

$$f(x) = (-\Delta)^{1/2} \sum_{k=0}^{m} D_{m,n}^k (N_m^k f)(x), \tag{6}$$

where the linear differential operator of order $m + k$

$$D_{m,n}^k : C^\infty(\mathbb{R}^n; S^m) \to C^\infty(\mathbb{R}^n; S^m)$$

is defined by

$$D_{m,n}^k = c_{m,n}^k \sum_{p=k}^{m} (n + 2m - 2p - 3)!!$$
$$\times \sum_{q=0}^{\min(p,m-p,p-k)} \frac{(-1)^q}{2^q q! (m-p-q)!(p-k-q)!} d^{p-q} i^q j^q j_x^{p-k-q} \operatorname{div}^k \tag{7}$$

with the coefficient

$$c_{m,n}^k = \frac{(-1)^k}{(k!)^2} \frac{2^{m-2} \, \Gamma\left(\frac{2m+n-1}{2}\right)}{\pi^{(n+1)/2} \, (n + 2m - 3)!!} \tag{8}$$

and the operators i, j, and j_x are defined in Section 2.1.

2.8. The Saint Venant operator. For integers m and r satisfying $0 \leq r \leq m$, let $S^{m-r} \otimes S^m$ be the space of $(2m - r)$-tensors on $\mathbb{R}^n$ which are symmetric in first $m - r$ and last m indices. *The Saint Venant operator* (2) *is defined by*

$$(W_m^r f)_{i_1 \ldots i_{m-r} j_1 \ldots j_m} = \sigma(i_1 \ldots i_{m-r}) \sigma(j_1 \cdots j_m) \sum_{l=0}^{m-r} (-1)^l \binom{m-r}{l}$$
$$\times \frac{\partial^{m-r} f_{i_1 \ldots i_{m-r-l} j_1 \ldots j_{r+l}}}{\partial x_{i_{m-r-l+1}} \ldots \partial x_{i_{m-r}} \partial x_{j_{r+l+1}} \ldots \partial x_{j_m}}. \tag{9}$$

In particular W_m^m is the identity operator.

2.9. The main result.

Theorem 2. *Let $0 \leq r \leq m$ and $n \geq 2$ be integers. For $f \in \mathcal{S}(\mathbb{R}^n; S^m)$, the tensor field $W_m^r f$ is recovered from the data $(N_m^0 f, \ldots, N_m^r f)$ by the inversion formula*

$$W_m^r f = (-\Delta)^{1/2} W_m^r \sum_{k=0}^{r} D_{m,n}^k (N_m^k f),$$

where the linear differential operator $D_{m,n}^k$ is defined by (7).

Theorem 2 is a generalization of Theorem 1 since W_m^m is the identity operator. In the case of $r = 0$ Theorem 2 actually coincides with [8, Theorem 2.12.3].

The first step in the proof of Theorem 2 is as follows. Since W_m^r is a differential operator with constant coefficients, it commutes with $(-\Delta)^{1/2}$. Applying the operator W_m^r to the equality (6), we write the result in the form

$$W_m^r f = (-\Delta)^{1/2} W_m^r \sum_{k=0}^{r} D_{m,n}^k (N_m^k f) + (-\Delta)^{1/2} W_m^r \sum_{k=r+1}^{m} D_{m,n}^k (N_m^k f).$$

Thus, to prove Theorem 2, it suffices to demonstrate that

$$W_m^r D_{m,n}^k = 0 \quad \text{for} \quad 0 \le r < k \le m. \tag{10}$$

The proof of (10) is presented in the next section.

3   Proof of Theorem 2

Applying the Fourier transform to (9), we obtain

$$\widehat{W_m^r f} = i^{m-r} \widehat{W}_m^r \widehat{f},$$

where i is the imaginary unit and the purely algebraic operator

$$\widehat{W}_m^r = \widehat{W}_m^r(y) : S^m \to S^{m-r} \otimes S^m \quad (y \in \mathbb{R}^n)$$

is defined by

$$(\widehat{W}_m^r h)_{i_1 \dots i_{m-r} j_1 \dots j_m} = \sigma(i_1 \dots i_{m-r}) \sigma(j_1 \cdots j_m) \sum_{l=0}^{m-r} (-1)^l \binom{m-r}{l} \times$$

$$\times h_{i_1 \dots i_{m-r-l} j_1 \dots j_{r+l}} \, y_{i_{m-r-l+1}} \cdots y_{i_{m-r}} y_{j_{r+l+1}} \cdots y_{j_m}.$$

This can be written in the coordinate-free form

$$\langle \widehat{W}_m^r h, u \otimes v \rangle = \sum_{l=0}^{m-r} (-1)^l \binom{m-r}{l} \langle h, (j_y^l u)(j_y^{m-r-l} v) \rangle \tag{11}$$

$$\text{for } u \in S^{m-r} \text{ and } v \in S^m.$$

On the other hand, applying the Fourier transform to (10), we see that (10) is equivalent to the statement

$$\widehat{W}_m^r \widehat{D}_{m,n}^k = 0 \quad \text{for} \quad 0 \le r < k \le m, \tag{12}$$

where the operator $\widehat{D}_{m,n}^k$ is defined by

$$\widehat{D}_{m,n}^k = c_{m,n}^k \sum_{p=k}^{m} (-1)^p (n+2m-2p-3)!!$$

$$\times \sum_{q=0}^{\min(p, m-p, p-k)} \frac{1}{2^q \, q! (m-p-q)! (p-k-q)!} \, i_y^{p-q} i^q j^q \, \text{div}^{p-k-q} \, j_y^k, \tag{13}$$

see [4, formula (8.7)].

We will use only one property of the operator $\widehat{D}_{m,n}^k$: as is seen from (13),

$$\widehat{D}_{m,n}^k = i_y^{r+1} B_{m,n}^k, \qquad \text{for } 0 \le r < k, \tag{14}$$

with some linear operator $B_{m,n}^k$. Therefore, to prove (12), it suffices to demonstrate that

$$\widehat{W}_m^r i_y^{r+1} = 0 \quad \text{for} \quad 0 \le r \le m-1. \tag{15}$$

By (11),

$$\langle \widehat{W}_m^r i_y^{r+1} h, u \otimes v \rangle = \sum_{l=0}^{m-r} (-1)^l \binom{m-r}{l} \langle i_y^{r+1} h, (j_y^l u)(j_y^{m-r-l} v) \rangle$$

$$= \Big\langle h, \sum_{l=0}^{m-r} (-1)^l \binom{m-r}{l} j_y^{r+1}\big((j_y^l u)(j_y^{m-r-l} v)\big) \Big\rangle.$$

This means that (15) holds for any $h \in S^{m-1}$ if and only if

$$\sum_{l=0}^{m-r} (-1)^l \binom{m-r}{l} j_y^{r+1}\big((j_y^l u)(j_y^{m-r-l} v)\big) = 0 \tag{16}$$

$$\text{for any } u \in S^{m-r}, \ v \in S^m, \quad 0 \le r < m.$$

The left-hand side of (16) is homogeneous of degree $m+1$ in y. It suffices to prove (16) for a unit vector y. In what follows, $y \in \mathbb{R}^n$ is a fixed vector satisfying $|y| = 1$.

The complex vector space $S^m = S^m \mathbb{R}^n$ is generated by powers x^m ($x \in \mathbb{R}^n$). Therefore (16) is equivalent to the statement

$$\sum_{l=0}^{m-r} (-1)^l \binom{m-r}{l} j_y^{r+1}\big((j_y^l x^{m-r})(j_y^{m-r-l} z^m)\big) = 0$$

$$\text{for any } x, z \in \mathbb{R}^n, \quad 0 \le r < m. \tag{17}$$

Since $j_y^l x^{m-r} = \langle x, y \rangle^l x^{m-r-l}$ and $j_y^{m-r-l} z^m = \langle z, y \rangle^{m-r-l} z^{r+l}$, the latter statement can be written as

$$\sum_{l=0}^{m-r} (-1)^l \binom{m-r}{l} \langle x, y \rangle^l \langle z, y \rangle^{m-r-l} j_y^{r+1}(x^{m-r-l} z^{r+l}) = 0 \tag{18}$$

for any $x, z \in \mathbb{R}^n$ and $0 \le r < m$. The equality (18) holds in the case $\langle x, y \rangle = \langle z, y \rangle = 0$ since all summands on the left-hand side are equal to zero.

Next, we prove (18) in the case $\langle x, y \rangle = 0$ but $\langle z, y \rangle \ne 0$. In this case (18) looks as follows:

$$j_y^{r+1}(x^{m-r} z^r) = 0. \tag{19}$$

Let us write (19) in coordinates

$$y^{i_1} \dots y^{i_{r+1}} \sum_{\pi \in \Pi_m} x_{i_{\pi(1)}} \dots x_{i_{\pi(m-r)}} z_{i_{\pi(m-r+1)}} \dots z_{i_{\pi(m)}} = 0.$$

After pulling the factor $y^{i_1} \dots y^{i_{r+1}}$ inside the sum, every summand contain at least one factor of the form $y^k x_k = 0$. This proves (19).

Quite similarly (18) is proved in the case $\langle x, y \rangle \neq 0$ but $\langle z, y \rangle = 0$.

Now, we prove (18) in the general case when $\alpha = \langle x, y \rangle \neq 0$ and $\beta = \langle z, y \rangle \neq 0$. We represent vectors $x, z \in \mathbb{R}^n$ in the form

$$x = \alpha y + x', \ \langle x', y \rangle = 0; \quad z = \beta y + z', \ \langle z', y \rangle = 0.$$

From this

$$x^{m-r-l} z^{r+l} = (\alpha y + x')^{m-r-l} (\beta y + z')^{r+l}$$
$$= \sum_{p=0}^{m-r-l} \sum_{q=0}^{r+l} \binom{m-r-l}{p} \binom{r+l}{q} \alpha^{m-r-l-p} \beta^{r+l-q} y^{m-p-q} x'^p z'^q.$$

Substituting this expression into (18), we obtain (up to a factor $\alpha^{m-r} \beta^m$)

$$\sum_{l=0}^{m-r} \sum_{p=0}^{m-r-l} \sum_{q=0}^{r+l} (-1)^l \binom{m-r}{l} \binom{m-r-l}{p} \binom{r+l}{q} \alpha^{-p} \beta^{-q}$$
$$\cdot j_y^{r+1} \left(y^{m-p-q} x'^p z'^q \right) = 0,$$

Denoting $\tilde{x} = \alpha^{-1} x'$ and $\tilde{z} = \beta^{-1} z'$, this can be written in the form

$$\sum_{l=0}^{m-r} \sum_{p=0}^{m-r-l} \sum_{q=0}^{r+l} (-1)^l \binom{m-r}{l} \binom{m-r-l}{p} \binom{r+l}{q} j_y^{r+1} (y^{m-p-q} \tilde{x}^p \tilde{z}^q) = 0.$$

To simplify notations, we denote $\tilde{x}$ and $\tilde{z}$ again by x and z respectively. Thus, we have to prove the statement

$$\sum_{l=0}^{m-r} \sum_{p=0}^{m-r-l} \sum_{q=0}^{r+l} (-1)^l \binom{m-r}{l} \binom{m-r-l}{p} \binom{r+l}{q} j_y^{r+1} (y^{m-p-q} x^p z^q) = 0 \quad (20)$$

for $x, z \in y^\perp$ and $0 \leq r < m$.

Since the last factor $j_y^{r+1}(y^{m-p-q} x^p z^q)$ on the left-hand side of (20) is independent of l, it makes sense to change the order of summations. We first change the order of summations over l and p

$$\sum_{p=0}^{m-r} \sum_{l=0}^{m-r-p} \sum_{q=0}^{r+l} (-1)^l \binom{m-r}{l} \binom{m-r-l}{p} \binom{r+l}{q} j_y^{r+1} (y^{m-p-q} x^p z^q) = 0$$

and then change the order of summations over l and q

$$\sum_{p=0}^{m-r} \sum_{q=0}^{m-p} \sum_{l=\max(0,q-r)}^{m-r-p} (-1)^l \binom{m-r}{l} \binom{m-r-l}{p} \binom{r+l}{q}$$
$$\times j_y^{r+1} \left(y^{m-p-q} x^p z^q \right) = 0.$$

This can be written in the form

$$\sum_{p=0}^{m-r}\sum_{q=0}^{m-p} C(m,r,p,q)\, j_y^{r+1}(y^{m-p-q}x^p z^q) = 0 \quad (x,z \in y^\perp, 0 \le r < m), \quad (21)$$

where

$$C(m,r,p,q) = \sum_{l=\max(0,q-r)}^{m-r-p} (-1)^l \binom{m-r}{l}\binom{m-r-l}{p}\binom{r+l}{q}, \quad (22)$$

$$0 \le p \le m-r, \quad 0 \le q \le m-p.$$

From (21) and (22), for $x, z \in y^\perp$, we have

$$j_y^{r+1}(y^{m-p-q}x^p z^q) = 0 \quad \text{if} \quad p \ge 0, q \ge 0, p+q \le m, r+1 > m-p-q. \quad (23)$$

Indeed, writing in coordinates

$$(y^{m-p-q}x^p z^q)_{i_1\ldots i_m}$$

$$= \frac{1}{m!} \sum_{\pi \in \Pi_m} y_{i_{\pi(1)}} \cdots y_{i_{\pi(m-p-q)}} x_{i_{\pi(m-p-q+1)}} \cdots x_{i_{\pi(m-q)}} z_{i_{\pi(m-q+1)}} \cdots z_{i_{\pi(m)}},$$

we have

$$\left(j_y^{r+1}(y^{m-p-q}x^p z^q)\right)_{i_{m-r}\ldots i_m} = \frac{1}{m!} \sum_{\pi \in \Pi_m} y^{i_1} \cdots y^{i_{r+1}}\, y_{i_{\pi(1)}} \cdots y_{i_{\pi(m-p-q)}}$$

$$\times\, x_{i_{\pi(m-p-q+1)}} \cdots x_{i_{\pi(m-q)}} z_{i_{\pi(m-q+1)}} \cdots z_{i_{\pi(m)}}.$$

In the case of $r+1 > m-p-q$, every summand of the sum contains either a factor of the form $y^j x_j = 0$ or a factor of the form $y^j z_j = 0$.

In virtue of (23), the summation in (21) can be restricted to (p,q) satisfying

$$p \ge 0, \quad q \ge 0, \quad p+q \le m-r-1. \quad (24)$$

In particular, $r < m$ and $p \le m-r-1$. In other words, (21) is equivalent to the statement

$$\sum_{p=0}^{m-r-1}\sum_{q=0}^{m-r-p-1} C(m,r,p,q)\, j_y^{r+1}(y^{m-p-q}x^p z^q) = 0 \quad (x,z \in y^\perp, 0 \le r < m).$$

$$(25)$$

Lemma 1. *For integers m,r,p,q satisfying (24) and $0 \le r < m$, the following equality holds:*

$$\sum_{l=\max(0,q-r)}^{m-r-p} (-1)^l \binom{m-r}{l}\binom{m-r-l}{p}\binom{r+l}{q} = 0. \quad (26)$$

With the help of Lemma 1, we immediately complete the proof of Theorem 2. Indeed, by comparing (22) and (26), we observe that all coefficients

$$C(m,r,p,q)$$

participating in (25) are equal to zero. This proves (21). As shown earlier, (21) implies the statement of Theorem 2.

Proof of Lemma 1. We assume binomial coefficients $\binom{k}{p}$ to be defined for all integers k and p under the agreement

$$\binom{k}{p} = 0 \quad \text{if either } k < 0 \text{ or } p < 0 \text{ or } k < p.$$

Then

$$
\begin{aligned}
C(m,r,p,q) &= \sum_{l=\max(0,q-r)}^{m-r-p} (-1)^l \binom{m-r}{l}\binom{m-r-l}{p}\binom{r+l}{q} \\
&= \sum_{l=-\infty}^{\infty} (-1)^l \binom{m-r}{l}\binom{r+l}{q}\binom{m-r-l}{p}.
\end{aligned}
\tag{27}
$$

From [3, p. 10], we have for $0 < \varepsilon \ll 1$,

$$\binom{n}{k} = \frac{1}{2\pi i} \int\limits_{|z|=\varepsilon} \frac{(1+z)^n}{z^{k+1}}\, dz.$$

In particular,

$$\binom{r+l}{q} = \frac{1}{2\pi i} \int\limits_{|z|=\epsilon} \frac{(1+z)^{r+l}}{z^{q+1}}\, dz,$$

$$\binom{m-r-l}{p} = \frac{1}{2\pi i} \int\limits_{|w|=\epsilon} \frac{(1+w)^{m-r-l}}{w^{p+1}}\, dw.$$

With the help of these formulas, we transform (27) as follows:

$$C(m,r,p,q)$$

$$= -\frac{1}{(2\pi)^2} \int\limits_{|z|=\epsilon} \int\limits_{|w|=\epsilon} \frac{(1+z)^r(1+w)^{m-r}}{z^{q+1}w^{p+1}} \sum_{l=-\infty}^{\infty} (-1)^l \binom{m-r}{l}\left(\frac{1+z}{1+w}\right)^l dw\, dz$$

$$= -\frac{1}{(2\pi)^2} \int\limits_{|z|=\epsilon} \int\limits_{|w|=\epsilon} \frac{(1+z)^r(1+w)^{m-r}}{z^{q+1}w^{p+1}} \left(1 - \frac{1+z}{1+w}\right)^{m-r} dw\, dz$$

$$= -\frac{1}{(2\pi)^2} \int\limits_{|z|=\epsilon} \int\limits_{|w|=\epsilon} \frac{(1+z)^r(w-z)^{m-r}}{z^{q+1}w^{p+1}}\, dw\, dz$$

$$= -\frac{1}{(2\pi)^2} \int\limits_{|z|=\epsilon} \int\limits_{|w|=\epsilon} \frac{(1+z)^r}{z^{q+1}w^{p+1}} \sum_{l=-\infty}^{\infty} (-1)^l \binom{m-r}{l} z^l w^{m-r-l}\, dw\, dz.$$

We perform the integration with respect to w. By the Cauchy integral formula, the only summand that survives corresponds to $l = m - r - p$.

Thus,

$$C(m,r,p,q) = \frac{(-1)^{m-r-p}}{2\pi \mathrm{i}} \binom{m-r}{p} \int\limits_{|z|=\epsilon} (1+z)^r z^{m-r-p-q-1}\, dz.$$

The integrand is a holomorphic function if $p + q \leq m - r - 1$. Therefore $C(m,r,p,q) = 0$ if $p + q \leq m - r - 1$. $\qquad\square$

References

[1] S. Bhattacharyya, V.P. Krishnan, S.K. Sahoo., *Momentum ray transforms and a partial data inverse problem for a polyharmonic operator*, SIAM J. Math. Anal., **55**:4 (2023), 4000–4038. Zbl 1522.35574

[2] S. Bhattacharyya, K. Krupchyk, S. K. Sahoo, G. Uhlmann. *Inverse problems for third-order nonlinear perturbations of biharmonic operators*, Commun. Partial Differ. Equations, **50**:3 (2025) 407–440. Zbl 8015702

[3] G.P. Egorychev, *Integral representation and the computation of combinatorial sums*, Translations of Mathematical Monographs, **59**, Amer. Math. Soc., Providence, 1984. Zbl 0524.05001

[4] S.R. Jathar, M. Kar, V.P. Krishnan, V.A. Sharafutdinov, *Normal operators for momentum ray transforms, I: The inversion formula*, J. Fourier Anal. Appl., **30**:5 (2024), Paper No. 58. Zbl 7938420

[5] S.K. Sahoo, M. Salo. *The linearized Calderón problem for polyharmonic operators*, J. Differ. Equations, **360** (2023), 407–451. Zbl 1512.35678

[6] R.K. Mishra, S.K. Sahoo, *The generalized Saint Venant operator and integral moment transforms*, Proc. Am. Math. Soc., **151**:1 (2023), 189–199. Zbl 1516.44002

[7] R.K. Mishra, S.K. Sahoo, *Injectivity and range description of integral moment transforms over m-tensor fields in* $\mathbb{R}^n$, SIAM J. Math. Anal., **53**:1 (2021), 253–278. Zbl 1464.44002

[8] V.A. Sharafutdinov, *Integral geometry of tensor fields*, Inverse Ill-posed Probl. Ser., VSP, Utrecht, 1994. MR1374572

SHUBHAM R. JATHAR, MANAS KAR
INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH BHOPAL,
INDORE BY-PASS ROAD, BHOPAL 465066, MADHYA PRADESH, INDIA
Email address: shubham.jathar7@gmail.com, manas@iiserb.ac.in

VENKATESWARAN P. KRISHNAN
TATA INSTITUTE OF FUNDAMENTAL RESEARCH,
CENTRE FOR APPLICABLE MATHEMATICS, SHARADA NAGAR, BENGALURU 560065, INDIA
Email address: vkrishnan@tifrbng.res.in

VLADIMIR A. SHARAFUTDINOV
SOBOLEV INSTITUTE OF MATHEMATICS,
4 KOPTYUG AVE., 630090, NOVOSIBIRSK, RUSSIA
Email address: sharaf@math.nsc.ru