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THE IMPULSIVE KELVIN–VOIGT EQUATIONS FOR TWO-COMPONENT MIXTURES OF VISCOELASTIC FLUIDS

S.N. ANTONTSEV, I.V. KUZNETSOV, D.A. PROKUDIN, S.A. SAZHENKOV

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Abstract: We study the multidimensional initial-boundary value problem for the system of Kelvin–Voigt equations of a two-component mixture of viscoelastic fluids with nonlinear convective terms and a linear impulsive term — a regular minor term describing impulsive source or damping. The impulsive term depends on a positive integer parameter n and, as $n \to +\infty$, weakly* converges to an expression including the Dirac delta-function, which models impulsive source or damping at the initial moment of time. We prove that an infinitesimal initial impulsive layer, associated with the Dirac delta function, is formed as $n \to +\infty$, and that the family of regular weak solutions to the original problem converges to the strong solution of a two-scale microscopic-macroscopic model. This model consists of two initial-boundary value problems that should be solved successively: at first, the flow of the mixture is defined on the infinitesimal initial impulsive layer set at the microscopic ('fast') timescale, and, at second, the outer flow beyond the initial impulsive layer is defined at the macroscopic ('slow') timescale.

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The equations of the initial impulsive layer inherit the full information about the profile of the original non-instantaneous source or damping.

Keywords: Kelvin–Voigt equations, two-component mixture, impulsive partial differential equation, initial layer.

1 Introduction

We study the Cauchy–Dirichlet problem for the Kelvin–Voigt equations of a two-component mixture of viscoelastic fluids in the presence of impulsive phenomena:

$$\begin{cases} \partial_t \boldsymbol{v}_i^n + \operatorname{div}_x\big(\boldsymbol{v}_i^n \otimes \boldsymbol{v}_i^n\big) = -\nabla_x \pi_i^n + \displaystyle\sum_{j=1}^{2} \mu_{ij}\Delta_x \boldsymbol{v}_j^n + \varkappa_i \Delta_x \partial_t \boldsymbol{v}_i^n \\ \qquad\qquad\qquad\qquad + \gamma\varphi^n(t)\boldsymbol{v}_i^n \quad \text{in } Q_T, \\ \operatorname{div}_x \boldsymbol{v}_i^n = 0 \quad \text{in } Q_T, \\ \boldsymbol{v}_i^n(\cdot,0) = \boldsymbol{v}_{0i} \quad \text{in } \Omega, \\ \boldsymbol{v}_i^n = \mathbf{0} \quad \text{on } \Gamma_T. \qquad\qquad (i=1,2) \end{cases} \tag{1}$$

Let us recall that, for each component $i \in \{1,2\}$ in the mixture, $(1)_1$ is the balance of momentum equation, $(1)_2$ is the balance of mass equation (in other terms, the incompressibility condition), $(1)_3$ is the initial condition, and $(1)_4$ is the no-slip condition on the outer fixed boundary of the whole viscoelastic continuum.

In (1), $\Omega \subset \mathbb{R}^d_x$ is a fixed bounded domain occupied by the viscoelastic continuum (fluid), with boundary $\partial\Omega \in C^2$, $d = 2$ or $d = 3$ is the dimension of the space of physical positions $\boldsymbol{x}$, t is the time variable, $T = \text{const} > 0$ is an arbitrarily fixed moment of time, $Q_T = \Omega \times (0,T)$ is the space-time cylinder, $\Gamma_T = \partial\Omega \times (0,T)$ is the lateral surface of Q_T, $\boldsymbol{v}_i^n = \boldsymbol{v}_i^n(\boldsymbol{x},t)$ $(i = 1,2)$ are the sought velocity fields of the mixture components, $\pi_i^n = \pi_i^n(\boldsymbol{x},t)$ $(i = 1,2)$ are the sought pressure fields of the mixture components, γ is a given constant coefficient of intensity of impulse, which can be either positive or negative, μ_{ij} $(i,j = 1,2)$ and $\varkappa_i$ $(i = 1,2)$ are given constant kinematic viscosity and retardation coefficients, respectively. We assume that

$$\boldsymbol{M} \stackrel{\text{def}}{=} \big(\mu_{ij}\big)_{i,j=1}^{2} > 0, \quad \mu_{ij} = \mu_{ji}, \quad \varkappa_i > 0 \quad (i,j = 1,2), \tag{2}$$

where relation $\boldsymbol{M} = \big(\mu_{ij}\big)_{i,j=1}^{2} > 0$ means that matrix $\boldsymbol{M}$ is positive definite, i.e., there is a positive constant μ_* such that the inequality $\boldsymbol{M\zeta}\cdot\boldsymbol{\zeta} \geq \mu_*|\boldsymbol{\zeta}|^2$ holds for all $\boldsymbol{\zeta} \in \mathbb{R}^d$. Initial distributions $\boldsymbol{v}_{0i} = \boldsymbol{v}_{0i}(\boldsymbol{x})$ $(i = 1,2)$ of the velocity fields and the profile of the impulsive impact $\varphi^n = \varphi^n(t)$ are also given and satisfy the following conditions:

$$\boldsymbol{v}_{0i} \in W_0^{2,2}(\Omega)^d, \quad \operatorname{div}_x \boldsymbol{v}_{0i} = 0 \quad (i = 1,2) \quad \text{on } \Omega, \tag{3}$$

function $\varphi^n = \varphi^n(t)$ is defined for each natural n, $n \geq n_0 = \left[\dfrac{1}{T}\right] + 2$, by the formula

$$\varphi^n(t) = n\Phi(nt), \quad t \in [0,T], \tag{4}$$

where $\Phi = \Phi(\vartheta)$ is a nonnegative smooth function supported on segment $\{0 \leq \vartheta \leq 1\}$ and having the unit mean value:

$$\int\limits_0^1 \Phi(\vartheta)\, d\vartheta = 1. \tag{5}$$

The assumptions on the sequence $\{\varphi^n\}_{n=1,2,\ldots}$ mean that it approximates the Dirac delta-function in the sense that $\varphi^n(\cdot) \underset{n\to+\infty}{\longrightarrow} \delta_{(t=0)}$ weakly* in $\mathcal{M}(0,T)$, where $\mathcal{M}(0,T)$ is the space of the Radon measures defined as the dual space of $C[0,T]$. Furthermore,

$$\int\limits_0^t \varphi^n(s)ds \leq 1 \quad \text{for } t \in [0,T], \quad \int\limits_0^T \varphi^n(s)ds = 1. \tag{6}$$

For Φ one may take the classical Friedrichs mollifier supported on $[0,1]$.

System (1) is a generalization to the two-component case of the well-known Kelvin–Voigt system of equations of dynamics of a one-component viscoelastic fluid. Note that equations $(1)_1$ $(i = 1,2)$ have two features, the presence of which mainly determines the novelty and originality of our research. The first of the features is the presence of the terms $\sum\limits_{j=1}^{2} \mu_{ij}\Delta_x \boldsymbol{v}_j^n$: unlike the one-component case, where the viscosity coefficient is a scalar, in the two-component case the viscosity coefficients μ_{ij} $(i,j = 1,2)$ form matrix $\boldsymbol{M}$ whose elements characterize viscous friction. More certainly, the diagonal elements of matrix $\boldsymbol{M}$ are responsible for the viscous friction within each component, and the non-diagonal elements are responsible for the viscous friction between the two components. In cases when $\boldsymbol{M}$ is a diagonal matrix, the subsystem of two equations $(1)_1$ decouples and, therefore, the whole system (1) falls into the well-known theory of the Kelvin–Voigt equations of dynamics of one-component viscoelastic fluids, see in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. In the present article, we consider a much more complicated case of a non-diagonal and non-triangular matrix $\mathbf{M}$ and do not impose any simplifying assumptions on it except the standard requirement of positive definiteness and symmetry. The study of solvability topics for hydrodynamic equations with non-diagonal viscosity matrices has a fairly notable history. Its foundations and main results can be found in [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. The peculiarity of this particular research is the presence of the impulsive terms $\gamma\varphi^n(t)\boldsymbol{v}_i^n$ $(i = 1,2)$.

From the mathematical point of view, this presence attracts significant interest, which is as follows. If we formally substitute $\boldsymbol{v}_i^n$ by $\boldsymbol{v}_i$ and the impulsive term by expression $\gamma\delta_{(t=0)}\boldsymbol{v}_i$ in $(1)_1$ and set $\boldsymbol{v}_i(\boldsymbol{x},0)=\boldsymbol{v}_{0i}(\boldsymbol{x})$ in Ω instead of $(1)_3$, then from the subsystem $(1)_{1,3}$ we deduce the subsystem, which is in the sense of distributions equivalent to the subsystem consisting of two equations

$$\partial_t\boldsymbol{v}_i+\operatorname{div}_x\big(\boldsymbol{v}_i\otimes\boldsymbol{v}_i\big)=-\nabla_x\pi_i+\sum_{j=1}^{2}\mu_{ij}\Delta_x\boldsymbol{v}_j+\varkappa_i\Delta_x\partial_t\boldsymbol{v}_i,\quad i=1,2,\tag{7}$$

and the 'corrected' initial conditions

$$\boldsymbol{v}_i(\boldsymbol{x},0)=(1+\gamma)\boldsymbol{v}_{0i}(\boldsymbol{x}),\quad i=1,2.\tag{8}$$

(Obviously, the form of subsystem $(1)_{2,4}$ remains intact.) However, the numerous observations in the theory of impulsive ordinary and partial differential equations signal that (7) and (8) most likely may not be the correct limit form of $(1)_{1,3}$ as $n\to+\infty$, see, for example [33, 34, 35, 36, 37, 38, 39]. Therefore, a careful and mathematically rigorous analysis of the limiting passage as $n\to+\infty$ is necessary in order to establish the correct limit mode when modeling a short impulsive action by an instantaneous one.

From the physical viewpoint, the impulsive term $\gamma\varphi^n(t)\boldsymbol{v}_i^n$ in $(1)_1$ is related to dilatant (shear-thickening) and pseudoplastic (shear-thinning) fluids [40, 41, 42]. Generally, incompressible dilatant and pseudoplastic fluids are fluids in which viscosity and velocity change significantly under impulsive loads. In this paper, we constrain ourselves to the case when the velocities change drastically, but the viscosity values remain constant. In view of this, absorption (in other terms, the velocity damping) occurs in dilatant liquids, which corresponds to a negative value of the coefficient γ. In turn, a sharp increase in velocity occurs in pseudoplastic liquids, i.e. the coefficient γ is positive. In applications, subsystem $(1)_{1,2}$ can be used, for example, for modeling of inhomogeneous loose media, since it is known that under the action of impulsive loading, a loose medium exhibits hydrodynamic properties. Indeed, when exposed to seismic shock waves, liquefaction of soils occurs, which leads to collapse of buildings [43, Part 1, Sec. 5].

Now, let us briefly describe the further organization of the article. In Section 2 we provide the rigorous formulation of the main results, which are the results of solvability for any fixed $n\geq n_0$ and passing to the limit as $n\to+\infty$. Sections 3 and 4 are devoted to justification of the main results.

2 The main results

2.1. Solvability of problem (1) for fixed $n\in\mathbb{N}$. Solution of problem (1) is understood in the weak sense. In order to formulate a definition of weak solution and for further considerations, we introduce the following functional spaces, which are widely used in mathematical theory of fluids:

$$\boldsymbol{\mathcal{V}}:=\big\{\boldsymbol{v}\in C_0^\infty(\Omega)^d:\ \operatorname{div}_x\boldsymbol{v}=0\big\},$$

$$\boldsymbol{H} := \text{ the closure of } \boldsymbol{\mathcal{V}} \text{ in the norm of } L^2(\Omega)^d,$$
$$\boldsymbol{V}^l := \text{ the closure of } \boldsymbol{\mathcal{V}} \text{ in the norm of } W^{l,2}(\Omega)^d, \quad l = 1, 2.$$

In the case $l = 1$, we denote $\boldsymbol{V}^1$ simply by $\boldsymbol{V}$. Note that condition (3) can be rewritten in the equivalent form $\boldsymbol{v}_{0i} \in \boldsymbol{V}^2$ $(i = 1, 2)$.

Now, for each $n \geq n_0$ $(n \in \mathbb{N})$, we introduce the notion of *regular* weak solutions to problem (1) as follows.

Definiton 1. *We say that a pair of vector-functions*

$$(\boldsymbol{v}_1^n, \boldsymbol{v}_2^n) = (\boldsymbol{v}_1^n(\boldsymbol{x}, t), \boldsymbol{v}_2^n(\boldsymbol{x}, t))$$

is a regular weak solution *to problem* (1)*, if it satisfies*

1) *the regularity requirements*

$$\boldsymbol{v}_i^n \in L^\infty(0, T; \boldsymbol{V}^2), \quad \partial_t \boldsymbol{v}_i^n \in L^2(0, T; \boldsymbol{V}) \quad (i = 1, 2),$$

2) *the integral equalities*

$$\int_0^s \int_\Omega \Big(\partial_t \boldsymbol{v}_i^n \cdot \boldsymbol{\phi}_i + \operatorname{div}_x(\boldsymbol{v}_i^n \otimes \boldsymbol{v}_i^n) \cdot \boldsymbol{\phi}_i + \varkappa_i \nabla_x \partial_t \boldsymbol{v}_i^n : \nabla_x \boldsymbol{\phi}_i + \sum_{j=1}^2 \mu_{ij} \nabla_x \boldsymbol{v}_j^n : \nabla_x \boldsymbol{\phi}_i \Big) \, d\boldsymbol{x} dt$$
$$= \gamma \int_0^s \varphi^n(t) \int_\Omega \boldsymbol{v}_i^n \cdot \boldsymbol{\phi}_i d\boldsymbol{x} dt \quad (i = 1, 2) \tag{9}$$

for any $s \in (0, T]$ and all pairs of test vector-functions $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ such that $\boldsymbol{\phi}_i \in L^\infty(0, T; \boldsymbol{V}^2)$, $\partial_t \boldsymbol{\phi}_i \in L^2(0, T; \boldsymbol{V})$,

3) *the initial conditions* $(1)_3$ *in the sense of the strong trace in $\boldsymbol{H}$, i.e.,*

$$\|\boldsymbol{v}_i^n(\cdot, t) - \boldsymbol{v}_{0i}(\cdot)\|_{\boldsymbol{H}} \underset{t \to 0+}{\longrightarrow} 0 \quad (i = 1, 2). \tag{10}$$

The first part of the main results of this article deals with the questions of global in time existence and uniqueness of solutions to problem (1) for each fixed $n \geq n_0$:

Theorem 1. *Assume that the input data for problem* (1) *meet the requirements set out in Section 1. Then problem* (1) *has a unique solution in the sense of Definition 1. Moreover, the solution satisfies the estimates*

$$\sum_{i=1}^2 \|\boldsymbol{v}_i^n\|_{L^\infty(0,T;\boldsymbol{V}^2)} \leq C_0, \tag{11}$$
$$\sum_{i=1}^2 \big(\|\partial_t \boldsymbol{v}_i^n\|_{L^1(0,T;\boldsymbol{H})} + \|\nabla_x \partial_t \boldsymbol{v}_i^n\|_{L^1(0,T;L^2(\Omega)^{d\times d})} \big) \leq C_0, \tag{12}$$

$$\sum_{i=1}^{2}\big(\|\partial_t \boldsymbol{v}_i^n\|_{L^2(0,T;\boldsymbol{H})} + \|\nabla_x \partial_t \boldsymbol{v}_i^n\|_{L^2(Q_T)^{d\times d}}\big) \le C_0\Big(T + \int\limits_0^T (\varphi^n(t))^2\, dt\Big), \tag{13}$$

with a constant C_0 independent of n.

Note that estimates (11) and (12) are uniform in n, but estimate (13) is not uniform in n, since $\int\limits_0^T (\varphi^n(t))^2\, dt \underset{n\to+\infty}{\longrightarrow} +\infty$ due to (4) and (5).

The proof of Theorem 1 is given further in Section 3.

2.2. Passage to the limit in problem (1), as $n \to +\infty$. The second part of the main results of the article deals with the passage to the limit in problem (1), as $n \to +\infty$:

Theorem 2. *Assume that the input data for problem (1) meet the requirements set out in Section 1. Let $\big\{(\boldsymbol{v}_1^n, \boldsymbol{v}_2^n)\big\}_{n\ge n_0}$ be the family of regular weak solutions to problem (1) in the sense of Definition 1.*

Then the following assertions hold true.

1. *The family $\big\{(\boldsymbol{v}_1^n, \boldsymbol{v}_2^n)\big\}_{n\ge n_0}$ is relatively compact in $L^2(0,T;\boldsymbol{V})^2$ and relatively weakly* compact in $L^\infty(0,T;\boldsymbol{V}^2)^2$, as $n\to\infty$: there exist a subsequence from $\big\{(\boldsymbol{v}_1^n, \boldsymbol{v}_2^n)\big\}_{n\ge n_0}$, still labeled by n, and a limit pair of vector-functions $(\boldsymbol{v}_1, \boldsymbol{v}_2) \in L^\infty(0,T;\boldsymbol{V}^2)^2$ such that*

$$\boldsymbol{v}_i^n \underset{n\to+\infty}{\longrightarrow} \boldsymbol{v}_i \text{ strongly in } L^2(0,T;\boldsymbol{V}), \quad \text{and weakly}^\star \text{ in } L^\infty(0,T;\boldsymbol{V}^2) \quad (i=1,2). \tag{14}$$

2. *The family of rescaled solutions $\big\{(\overline{\boldsymbol{v}}_1^n, \overline{\boldsymbol{v}}_2^n)\big\}_{n\ge n_0}$, $\overline{\boldsymbol{v}}_i^n\colon \Omega\times[0,1]\mapsto\mathbb{R}^d$ $(i=1,2)$, defined by the formula*

$$\overline{\boldsymbol{v}}_i^n(\boldsymbol{x},\vartheta) \overset{\text{def}}{=} \boldsymbol{v}_i^n\Big(\boldsymbol{x}, \frac{\vartheta}{n}\Big), \quad \vartheta\in[0,1] \quad (i=1,2), \tag{15}$$

is relatively compact in $L^2(0,1;\boldsymbol{V})$ and relatively weakly* compact in $L^\infty(0,1;\boldsymbol{V}^2)$: there exist a subsequence from $\big\{(\overline{\boldsymbol{v}}_1^n, \overline{\boldsymbol{v}}_2^n)\big\}_{n\ge n_0}$, still labeled by n, and a limit pair $(\overline{\boldsymbol{v}}_1, \overline{\boldsymbol{v}}_2) \in L^\infty(0,1;\boldsymbol{V}^2)^2$ such that

$$\overline{\boldsymbol{v}}_i^n \underset{n\to+\infty}{\longrightarrow} \overline{\boldsymbol{v}} \text{ strongly in } L^2(0,1;\boldsymbol{V}) \quad \text{and weakly}^\star \text{ in } L^\infty(0,1;\boldsymbol{V}^2) \quad (i=1,2). \tag{16}$$

3. *There exist two pairs of scalar functions $(\overline{\pi}_1, \overline{\pi}_2)$ and (π_1, π_2) such that the two pairs of the limit vector-functions $(\overline{\boldsymbol{v}}_1, \overline{\boldsymbol{v}}_2)$ and $(\boldsymbol{v}_1, \boldsymbol{v}_2)$ along with $(\overline{\pi}_1, \overline{\pi}_2)$ and (π_1, π_2) are a strong solution of the two Cauchy–Dirichlet problems that should be solved successively:*

3a. *Firstly, find the quadruple* $(\overline{\boldsymbol{v}}_1, \overline{\boldsymbol{v}}_2, \overline{\pi}_1, \overline{\pi}_2)$ *satisfying*

$$\begin{cases} \partial_\vartheta \overline{\boldsymbol{v}}_i = \varkappa_i \Delta_x \partial_\vartheta \overline{\boldsymbol{v}}_i + \gamma \Phi(\vartheta) \overline{\boldsymbol{v}}_i - \nabla_x \overline{\pi}_i \quad \textit{in } \Omega \times (0,1), \\ \operatorname{div}_x \overline{\boldsymbol{v}}_i = 0 \quad \textit{in } \Omega \times (0,1), \\ \overline{\boldsymbol{v}}_i(\cdot, 0) = \boldsymbol{v}_{0i} \quad \textit{in } \Omega, \\ \overline{\boldsymbol{v}}_i = \mathbf{0} \quad \textit{on } \partial\Omega \times (0,1) \qquad (i = 1, 2). \end{cases} \tag{17}$$

3b. *Secondly, find the quadruple* $(\boldsymbol{v}_1, \boldsymbol{v}_2, \pi_1, \pi_2)$ *satisfying*

$$\begin{cases} \partial_t \boldsymbol{v}_i + \operatorname{div}_x \big(\boldsymbol{v}_i \otimes \boldsymbol{v}_i \big) \\ \qquad = \operatorname{div}_x \Big(\sum_{j=1}^{2} \mu_{ij} \nabla_x \boldsymbol{v}_j + \varkappa_i \nabla_x \partial_t \boldsymbol{v}_i \Big) - \nabla_x \pi_i \quad \textit{in } Q_T, \\ \operatorname{div}_x \boldsymbol{v}_i = 0 \quad \textit{in } Q_T, \\ \boldsymbol{v}_i(\cdot, 0) = \overline{\boldsymbol{v}}_i(\cdot, 1) \quad \textit{in } \Omega, \\ \boldsymbol{v}_i = \mathbf{0} \quad \textit{on } \Gamma_T \qquad (i = 1, 2), \end{cases} \tag{18}$$

where the pair of initial vector-functions $\big(\boldsymbol{v}_1(\cdot,0), \boldsymbol{v}_2(\cdot,0)\big)$ *is defined by the solution of problem* (17) *at the moment* $\vartheta = 1$.

We call equations $(17)_{1,2}$ *the initial infinitesimal layer equations.* Equations $(17)_1$ contain function $\Phi(\vartheta)$ and therefore preserve the whole information about the instantaneous impulsive profile. Due to rescaling $t = \vartheta/n$ (see in (15)), the independent variable ϑ can be regarded to as the *fast time* variable and quadruple $(\overline{\boldsymbol{v}}_1, \overline{\boldsymbol{v}}_2, \overline{\pi}_1, \overline{\pi}_2)$ can be called a *microscopic initial layer solution*, while t is the *slow time* and quadruple $(\boldsymbol{v}_1, \boldsymbol{v}_2, \pi_1, \pi_2)$ is a *macroscopic outer solution.* Thus, (17)–(18) is the two-scale microscopic-macroscopic problem. Conditions $(18)_3$ may be fairly interpreted as the *interfacial* conditions between the initial microscopic impulsive layer and the macroscopic outer flow.

The notions of strong solutions to problem (17)–(18) is as follows.

Definiton 2. *The set consisting of eight functions* $(\overline{\boldsymbol{v}}_i, \overline{\pi}_i, \boldsymbol{v}_i, \pi_i)$ $(i = 1, 2)$ *is called a* strong solution *to problem* (17)–(18)*, if*

(I) (ia) *the quadruple of functions* $(\overline{\boldsymbol{v}}_i, \overline{\pi}_i)$ $(i = 1, 2)$ *satisfies the regularity conditions*

$$\begin{aligned} &\overline{\boldsymbol{v}}_i \in C([0,1]; \boldsymbol{H}) \cap L^\infty(0,1; \boldsymbol{V}^2), \quad \partial_\vartheta \overline{\boldsymbol{v}}_i, \, \Delta_x \partial_\vartheta \overline{\boldsymbol{v}}_i \in L^\infty(0,1; \boldsymbol{H}), \\ &\nabla_x \overline{\pi}_i \in L^2(\Omega \times (0,1))^d, \end{aligned} \tag{19a}$$

(ib) *equations* $(17)_1$ *and* $(17)_2$ *hold a.e. in* $\Omega \times (0,1)$*,*

(ic) *the initial conditions* $(17)_3$ *hold in the strong trace sense in* $\boldsymbol{H}$*, i.e.*

$$\|\overline{\boldsymbol{v}}_i(\cdot, \vartheta) - \boldsymbol{v}_{0i}(\cdot)\|_{\boldsymbol{H}} \underset{\vartheta \to 0+}{\longrightarrow} 0; \tag{19b}$$

(II)(iia) *the quadruple of functions* $(\boldsymbol{v}_i, \pi_i)$ *($i = 1, 2$) satisfies the regularity conditions*

$$\begin{aligned}&\boldsymbol{v}_i \in C([0,T];\boldsymbol{H}) \cap L^\infty(0,T;\boldsymbol{V}^2), \quad \partial_t \boldsymbol{v}_i,\ \Delta_x \partial_t \boldsymbol{v}_i \in L^\infty(0,T;\boldsymbol{H}),\\ &\nabla_x \pi_i \in L^2(Q_T)^d,\end{aligned} \tag{20a}$$

(iib) *equations* $(18)_1$ *and* $(18)_2$ *hold a.e. in* Q_T,

(iic) *the initial conditions* $(18)_3$ *hold in the strong trace sense in* $\boldsymbol{H}$, *i.e.*

$$\|\boldsymbol{v}_i(\cdot,t) - \overline{\boldsymbol{v}}_i(\cdot,1)\|_{\boldsymbol{H}} \underset{t\to 0+}{\longrightarrow} 0. \tag{20b}$$

The proof of Theorem 2 is given further in Section 4.

3 Proof of Theorem 1

The proof of Theorem 1 is split into several steps. In Section 3.1 we introduce the finite-dimensional Galerkin's approximations $\big(\boldsymbol{v}_1^{m,n}, \boldsymbol{v}_2^{m,n}\big)$ ($m \in \mathbb{N}$) for the sought regular weak solution $\big(\boldsymbol{v}_1^n, \boldsymbol{v}_2^n\big)$. The super-index m indicates the dimension of the subspace the approximation $\boldsymbol{v}_i^{m,n}$ ($i = 1, 2$) belongs to. In Sections 3.2–3.4 we derive uniform estimates for Galerkin's approximations. In Section 3.5 these estimates are used for justification of the limiting passage as $m \to +\infty$, which provides existence of a weak regular solution to problem (1) and the uniform estimates in n on the family $\big\{\big(\boldsymbol{v}_1^n, \boldsymbol{v}_2^n\big)\big\}_{n\ge n_0}$. Section 3.6 is devoted to justification of the uniqueness of the regular weak solution.

3.1. Galerkin's approximations. Following [44, Ch. II, § 4], [45, § 3], we introduce the orthogonal basis $\{\boldsymbol{\psi}_k\}_{k=1}^\infty$ in $\boldsymbol{V}^2$, which is also an orthonormal basis in $\boldsymbol{H}$, consisting of the eigenfunctions of the spectral problem

$$\int\limits_\Omega \nabla_x \boldsymbol{\psi}_k : \nabla_x \boldsymbol{\Phi}\, d\boldsymbol{x} = \lambda_k \int\limits_\Omega \boldsymbol{\psi}_k \cdot \boldsymbol{\Phi}\, d\boldsymbol{x}, \quad \forall\, \boldsymbol{\Phi} \in \boldsymbol{V}^2, \quad k = 1, 2, \dots. \tag{21}$$

Here $\{\lambda_k\}_{k=1,2,\dots}$ is the sequence of the positive eigenvalues.

We construct a regular weak solution $\big(\boldsymbol{v}_1^n, \boldsymbol{v}_2^n\big)$ to problem (1) as a limit of the sequence of the finite-dimensional Galerkin's approximations

$$\boldsymbol{v}_i^n = \lim_{m\to+\infty} \boldsymbol{v}_i^{m,n} \quad (i = 1, 2), \tag{22a}$$

where

$$\boldsymbol{v}_i^{m,n}(\cdot,t) = \sum_{s=1}^m v_{s,i}^{m,n}(t) \boldsymbol{\psi}_s(\cdot), \quad t \in [0,T] \quad (i = 1, 2). \tag{22b}$$

Unknown coefficients $v_{k,i}^{m,n}(t)$ ($k = 1, 2, \dots, m$, $i = 1, 2$) are defined as the solutions of Galerkin's system of $2m$ ordinary differential equations:

$$(1 + \varkappa_i \lambda_k) \frac{dv_{k,i}^{m,n}(t)}{dt} + \int\limits_\Omega \mathrm{div}_x \big(\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t) \otimes \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)\big) \cdot \boldsymbol{\psi}_k(\boldsymbol{x}) d\boldsymbol{x}$$

$$+\int_\Omega \Big(\sum_{j=1}^{2}\mu_{ij}\nabla_x \boldsymbol{v}_j^{m,n}(\boldsymbol{x},t):\nabla_x\boldsymbol{\psi}_k(\boldsymbol{x})-\gamma\varphi^n(t)\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)\cdot\boldsymbol{\psi}_k(\boldsymbol{x})\Big)d\boldsymbol{x}=0,$$
$$k=1,2,\dots,m \quad (i=1,2), \tag{23}$$

endowed with the initial conditions

$$v_{k,i}^{m,n}(0)=v_{0,k,i},\quad k=1,2,\dots,m \quad (i=1,2), \tag{24}$$

where constants $v_{0,k,i}$ are the Fourier coefficients of $\boldsymbol{v}_{0i}$ in the basis $\{\boldsymbol{\psi}_k\}_{k\in\mathbb{N}}$. We have

$$v_{0,k,i}=\int_\Omega \boldsymbol{v}_{0i}\cdot\boldsymbol{\psi}_k\,d\boldsymbol{x},\quad \boldsymbol{v}_{0i}^m=\sum_{k=1}^{m}v_{0,k,i}\boldsymbol{\psi}_k \underset{m\to+\infty}{\longrightarrow} \boldsymbol{v}_{0i}\quad\text{strongly in } \boldsymbol{V}^2 \quad (i=1,2).$$

Note that the coefficients $v_{0,k,i}$ and the approximate initial vector-function $\boldsymbol{v}_{0i}^m$ do not depend on n.

Since $1+\varkappa_i\lambda_k>1$ $(k=1,\dots,m,\ i=1,2)$, by Peano theorem, system (23), (24) has a solution $v_{k,i}^{m,n}(t)$ $(i=1,2)$ for each $m\in\mathbb{N}$ on some interval $(0,T^{mn})$. Accordingly, the approximate solution $\big(\boldsymbol{v}_1^{m,n},\boldsymbol{v}_2^{m,n}\big)$ exists in the space-time cylinder $\Omega\times(0,T^{mn})$.

3.2. The first energy estimate and continuation of $\big(\boldsymbol{v}_1^{m,n},\boldsymbol{v}_2^{m,n}\big)$.

Lemma 1. *Assume that the input data for problem (1) meet the requirements set out in Section 1. Let $n\in\mathbb{N}$ be arbitrarily fixed. Then each vector-function of the sequence $\big\{\big(\boldsymbol{v}_1^{m,n},\boldsymbol{v}_2^{m,n}\big)\big\}_{m=1,2,\dots}$ can be continued from $(0,T^{mn})$ onto the whole interval $(0,T]$ and satisfies the first energy estimate*

$$\sum_{i=1}^{2}\Big(\operatorname*{ess\,sup}_{t\in(0,T)}\|\boldsymbol{v}_i^{m,n}(\cdot,t)\|_{\boldsymbol{H}}+\operatorname*{ess\,sup}_{t\in(0,T)}\|\nabla_x\boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)^{d\times d}}\Big)+\sum_{i=1}^{2}\|\nabla_x\boldsymbol{v}_i^{m,n}\|_{L^2(Q_T)^{d\times d}}$$
$$\le M_0\Big(1+\sum_{i=1}^{2}\|\boldsymbol{v}_{0i}\|_{\boldsymbol{H}}+\sum_{i=1}^{2}\|\nabla_x\boldsymbol{v}_{0i}\|_{L^2(\Omega)^{d\times d}}\Big), \tag{25}$$

where the constant M_0 depends on T, Ω, d, μ_{ij}, $\varkappa_i$, and γ and is independent of m and n.

Proof is based on multiplying the (k,i)-th equation in system (23) by $v_{k,i}^{m,n}$, followed by summation over k from 1 to m and over i from 1 to 2 and additional account of relations (6) and (21). In principle, it is carried out in the same way as the derivation of the first energy estimate and the justification of continuation of solution in [45, proof of Th. 3.2]. □

3.3. The second energy estimate.

Lemma 2. *Assume that the input data for problem (1) meet the requirements set out in Section 1. Let $n \in \mathbb{N}$ be arbitrarily fixed. Then the family of Galerkin's approximations $\big(\boldsymbol{v}_1^{m,n}, \boldsymbol{v}_2^{m,n}\big)$ satisfies the second energy estimate*

$$\sum_{i=1}^{2}\Big(\operatorname*{ess\,sup}_{t\in(0,T)}\|\nabla_x \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)^{d\times d}} + \operatorname*{ess\,sup}_{t\in(0,T)}\|\Delta_x \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{\boldsymbol{H}}\Big) + \sum_{i=1}^{2}\|\Delta_x \boldsymbol{v}_i^{m,n}\|_{L^2(0,T;\boldsymbol{H})}$$
$$\le M_1\Big(1+\sum_{i=1}^{2}\|\boldsymbol{v}_{0i}\|_{\boldsymbol{H}} + \sum_{i=1}^{2}\|\nabla_x \boldsymbol{v}_{0i}\|_{L^2(\Omega)^{d\times d}} + \sum_{i=1}^{2}\|\Delta_x \boldsymbol{v}_{0i}\|_{\boldsymbol{H}}\Big), \qquad (26)$$

where the constant M_1 depends on T, Ω, d, μ_{ij}, $\varkappa_i$, and γ and is independent of m and n.

Proof. We multiply the (k,i)-th equation in (23) by $\lambda_k v_{k,i}^{m,n}(t)$ and sum up the resulting equations over k from 1 till m and over i from 1 till 2. Thus, we arrive at the integral equality

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{2}\Big(\|\nabla_x \boldsymbol{v}_i^{m,n}(\cdot,t)\|^2_{L^2(\Omega)^{d\times d}} + \varkappa_i\|\Delta_x \boldsymbol{v}_i^{m,n}(\cdot,t)\|^2_{\boldsymbol{H}}\Big)$$
$$+\sum_{i,j=1}^{2}\mu_{ij}\int_\Omega \Delta_x \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)\cdot\Delta_x \boldsymbol{v}_j^{m,n}(\boldsymbol{x},t)d\boldsymbol{x}$$
$$=\sum_{i=1}^{2}\int_\Omega \operatorname{div}_x\big(\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)\otimes \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)\big)\cdot\Delta_x \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)d\boldsymbol{x}$$
$$-\gamma\varphi^n(t)\sum_{i=1}^{2}\int_\Omega \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)\cdot\Delta_x \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)d\boldsymbol{x}. \qquad (27)$$

We integrate the second term in the right hand side by parts to get

$$-\gamma\varphi^n(t)\sum_{i=1}^{2}\int_\Omega \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)\cdot\Delta_x \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)d\boldsymbol{x} = \gamma\varphi^n(t)\sum_{i=1}^{2}\int_\Omega |\nabla_x \boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)|^2 d\boldsymbol{x}.$$

Using the Sobolev embedding inequality [46, Ch. I, § 8]

$$\|\boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^4(\Omega)^d} \le M_2(d,\Omega)\|\nabla_x \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)^{d\times d}} \quad (i=1,2), \qquad (28)$$

the combination of (28) with the well-known embedding inequality that allows to estimate the norm in $\boldsymbol{V}^2$ by the norm of Laplacian in $\boldsymbol{H}$ (see [44, Ch. I, § 1, Lem. 4]):

$$\|\nabla_x \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^4(\Omega)^{d\times d}} \le M_3(d,\Omega)\|\Delta_x \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{\boldsymbol{H}} \quad (i=1,2), \qquad (29)$$

Hölder's, Cauchy's and Young's inequalities, and the first energy inequality, we estimate

$$\left|\sum_{i=1}^{2}\int_{\Omega}\operatorname{div}_x\big(\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)\otimes\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)\big)\cdot\Delta_x\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)\,d\boldsymbol{x}\right|$$

$$\leq \frac{1}{2}\sum_{i=1}^{2}\big\|\operatorname{div}_x\big(\boldsymbol{v}_i^{m,n}\otimes\boldsymbol{v}_i^{m,n}\big)(\cdot,t)\big\|_{\boldsymbol{H}}^2+\frac{1}{2}\sum_{i=1}^{2}\|\Delta_x\boldsymbol{v}_i^{m,n}(\cdot,t)\|_{\boldsymbol{H}}^2 \tag{30}$$

and, further,

$$\begin{aligned}\big\|\operatorname{div}_x\big(\boldsymbol{v}_i^{m,n}\otimes\boldsymbol{v}_i^{m,n}\big)(\cdot,t)\big\|_{\boldsymbol{H}}^2 &\leq M_4(d)\|\boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^4(\Omega)^d}^2\|\nabla_x\boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^4(\Omega)^{d\times d}}^2\\ &\leq M_5\|\Delta_x\boldsymbol{v}_i^{m,n}(\cdot,t)\|_{\boldsymbol{H}}^2 \quad (i=1,2),\end{aligned} \tag{31}$$

where the constant M_5 depends on M_2, M_3, M_4, and the right hand side of the first energy inequality, and is independent of m and n.

Now, using (30) and (31), we can estimate the right hand side of (27) from above as follows:

$$\begin{aligned}&\sum_{i=1}^{2}\int_{\Omega}\operatorname{div}_x\big(\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)\otimes\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)\big)\cdot\Delta_x\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)d\boldsymbol{x}\\ &-\gamma\varphi^n(t)\sum_{i=1}^{2}\int_{\Omega}\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)\cdot\Delta_x\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)d\boldsymbol{x}\\ &\leq\left|\sum_{i=1}^{2}\int_{\Omega}\operatorname{div}_x\big(\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)\otimes\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)\big)\cdot\Delta_x\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)d\boldsymbol{x}\right|\\ &\qquad+|\gamma|\varphi^n(t)\sum_{i=1}^{2}\int_{\Omega}|\nabla_x\boldsymbol{v}_i^{m,n}(\boldsymbol{x},t)|^2d\boldsymbol{x}\\ &\leq M_5\sum_{i=1}^{2}\|\Delta_x\boldsymbol{v}_i^{m,n}(\cdot,t)\|_{\boldsymbol{H}}^2+|\gamma|\varphi^n(t)\sum_{i=1}^{2}\|\nabla_x\boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)^{d\times d}}^2.\end{aligned} \tag{32}$$

Introducing (for brevity)

$$Z^{m,n}(t):=\sum_{i=1}^{2}\big(\|\nabla_x\boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)^{d\times d}}^2+\varkappa_i\|\Delta_x\boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)^d}^2\big),$$

joining (27) and (32), and performing some simplest estimation, we arrive at the differential inequality

$$\frac{dZ^{m,n}(t)}{dt}+\mu_*\sum_{i=1}^{2}\|\Delta_x\boldsymbol{v}_i^{m,n}(\cdot,t)\|_{\boldsymbol{H}}^2\leq\big(M_5+|\gamma|\varphi^n(t)\big)Z^{m,n}(t). \tag{33}$$

Here recall that μ_* is a positive constant from the property of positive definiteness of matrix $\boldsymbol{M}$, see back in Section 1.

The second energy estimate (26) now readily follows from (33) by the Gronwall–Bellmann lemma and relations (6). Lemma 2 is proved. □

Corollary 1. *The family of Galerkin's approximations* $\big(\boldsymbol{v}_1^{m,n}, \boldsymbol{v}_2^{m,n}\big)$ *admits the uniform in* m *and* n *estimate*

$$\sum_{i=1}^{2} \|\boldsymbol{v}_i^{m,n}\|_{L^\infty(0,T;\boldsymbol{V}^2)} \le M_6\Big(\sum_{i=1}^{2}\|\boldsymbol{v}_{0i}\|_{\boldsymbol{H}} + \sum_{i=1}^{2}\|\nabla_x \boldsymbol{v}_{0i}\|_{L^2(\Omega)^{d\times d}} + \sum_{i=1}^{2}\|\Delta_x \boldsymbol{v}_{0i}\|_{\boldsymbol{H}} + 1\Big), \tag{34}$$

where the positive constant M_6 *depends only on* M_1*,* $M_3(d,\Omega)$*, and* d*.*

3.4. Uniform estimates for $\partial_t \boldsymbol{v}_i^{m,n}$ and $\nabla_x \partial_t \boldsymbol{v}_i^{m,n}$.

Lemma 3. *Assume that the input data for problem* (1) *meet the requirements set out in Section 1. Let* $n \in \mathbb{N}$ *be arbitrarily fixed. Then the family of Galerkin's approximations* $\big(\boldsymbol{v}_1^{m,n}, \boldsymbol{v}_2^{m,n}\big)$ *admits the following estimates:*

$$\sum_{i=1}^{2}\big(\|\partial_t \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{\boldsymbol{H}}^2 + \|\nabla_x \partial_t \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)^{d\times d}}^2\big) \le M_7 + M_8\,\gamma^2(\varphi^n(t))^2, \quad \forall\, t \in [0,T], \tag{35}$$

and

$$\sum_{i=1}^{2}\big(\|\partial_t \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{\boldsymbol{H}} + \|\nabla_x \partial_t \boldsymbol{v}_i^{m,n}(\cdot,t)\|_{L^2(\Omega)^{d\times d}}\big) \le 2\sqrt{2M_7} + 2\sqrt{2M_8}\,|\gamma|\varphi^n(t), \quad \forall\, t \in [0,T], \tag{36}$$

where the positive constants M_7 *and* M_8 *do not depend on* m *and* n*.*

Proof. Derivation of estimate (35) is based on multiplying the (k,i)-th equation in system (23) by $\dfrac{dv_{k,i}^{m,n}}{dt}$, followed by summation over k from 1 to m and over i from 1 to 2 and additional account of relations $(2)_{1,2}$ and (21). In principle, it is carried out in the same way as the derivation of the estimate (3.3) (in case $d = 3$) and estimate (3.3') (in case $d = 2$) from Theorem 3.2 in [45]. Estimate (36) readily follows from estimate (35) by means of the elementary inequality $A+B+C+D \le 2\sqrt{A^2+B^2+C^2+D^2}$, $\forall\, A,B,C,D \in \mathbb{R}$. □

3.5. Passage to the limit as $m \to \infty$. Uniform bounds in n for the sequence $\big\{\big(\boldsymbol{v}_1^n, \boldsymbol{v}_2^n\big)\big\}_{n\ge n_0}$.

By Corollary 1 and the Alaoglu theorem we conclude that

$$\text{the family } \big\{\big(\boldsymbol{v}_1^{m,n}, \boldsymbol{v}_2^{m,n}\big)\big\}_{m=1,2,\dots} \text{ is uniformly bounded in } L^\infty(0,T;\boldsymbol{V}^2)^2 \text{ for each fixed } n \ge n_0. \tag{37}$$

By Lemma 3 we conclude that

$$\text{the family } \left\{\left(\partial_t \boldsymbol{v}_1^{m,n}, \partial_t \boldsymbol{v}_2^{m,n}\right)\right\}_{m=1,2,\dots} \text{is uniformly bounded in } L^\infty(0,T;\boldsymbol{V})^2 \text{ for each fixed } n \geq n_0. \tag{38}$$

Thus, choosing a suitable subsequence, if necessary, we establish that there exist limit functions $\boldsymbol{v}_1^n$ and $\boldsymbol{v}_2^n$ such that the limiting relations

$$\boldsymbol{v}_i^{m,n} \underset{m\to+\infty}{\longrightarrow} \boldsymbol{v}_i^n \text{ weakly}^\star \text{ in } L^\infty(0,T;\boldsymbol{V}^2), \tag{39}$$

$$\partial_t \boldsymbol{v}_i^{m,n} \underset{m\to+\infty}{\longrightarrow} \partial_t \boldsymbol{v}_i^n \text{ weakly}^\star \text{ in } L^\infty(0,T;\boldsymbol{V}) \tag{40}$$

hold true for each $i = 1, 2$ and $n \geq n_0$.

Further note that, by the Rellich theorem, $\boldsymbol{V}^2$ is compactly embedded in $\boldsymbol{V}$. Due to this, by (37), (38) and the Aubin–Lions–Simon compactness theorem [47, Corol. 4], choosing one more subsequence, if necessary, we establish the limiting relation

$$\boldsymbol{v}_i^{m,n} \underset{m\to+\infty}{\longrightarrow} \boldsymbol{v}_i^n \quad \text{strongly in } C([0,T];\boldsymbol{V}) \tag{41}$$

for each $i = 1, 2$ and $n \geq n_0$.

Now, based on the limiting relations (39)–(41), the limiting transition in the Galerkin's system (22)–(24) as $m \to +\infty$ along the chosen subsequence is quite standard, and, similarly to [5] or [48, Ch. 6], as a result of the limiting transition we derive the integral equalities $(9)_{i=1,2}$ and the initial relations (10). Also, we note that the regularity requirements (1) for $\boldsymbol{v}_i^n$ in Definition 1 follow directly from (39)–(40) and by this we complete the justification of existence of a weak regular solution to problem (1).

Finally, passing to the limit as $m \to +\infty$ in the estimates (34)–(36) on the base of the compactness properties (37) and (38) and the well-known property of weak* lower semicontinuity for norms [49, Ch. 1, § 1.1.3; Ch. 2, § 2.3, Propos. 2.3.2], followed by additional integration of the resulting estimates for $\partial_t \boldsymbol{v}_n$ and $\nabla_x \partial_t \boldsymbol{v}_n$ in t on $(0, T)$, we deduce the estimates (11)–(13), with C_0 depending on M_6, M_7, M_8, γ, d, $\|\boldsymbol{v}_{0i}\|_{\boldsymbol{V}^2}$ $(i = 1, 2)$, and independent of n.

3.6. Uniqueness of the regular weak solution. Let $\left(\boldsymbol{v}_{1,1}^n, \boldsymbol{v}_{1,2}^n\right)$ and $\left(\boldsymbol{v}_{2,1}^n, \boldsymbol{v}_{2,2}^n\right)$ be two regular weak solutions to problem (1) corresponding to the same input data. Set $\boldsymbol{W}_i^n = \boldsymbol{v}_{1,i}^n - \boldsymbol{v}_{2,i}^n$ $(i = 1, 2)$.

Subtracting $(9)_i$ with $\left(\boldsymbol{v}_{2,1}^n, \boldsymbol{v}_{2,2}^n\right)$ from $(9)_i$ with $\left(\boldsymbol{v}_{1,1}^n, \boldsymbol{v}_{1,2}^n\right)$ for each $i = 1, 2$, taking $\boldsymbol{W}_i^n$ for the test functions $\boldsymbol{\phi}_i$ $(i = 1, 2)$ and, finally, summing up the results over $i = 1, 2$, we arrive at the following energy relation:

$$\frac{1}{2}\sum_{i=1}^{2} \frac{d}{dt}\left(\|\boldsymbol{W}_i^n(\cdot,t)\|_{\boldsymbol{H}}^2 + \varkappa_i \|\nabla_x \boldsymbol{W}_i^n(\cdot,t)\|_{L^2(\Omega)^{d\times d}}^2\right)$$

$$+ \sum_{i,j=1}^{2} \mu_{ij} \int_\Omega \nabla_x \boldsymbol{W}_i^n(\boldsymbol{x},t) : \nabla_x \boldsymbol{W}_j^n(\boldsymbol{x},t) d\boldsymbol{x}$$

$$= \gamma\varphi^n(t)\sum_{i=1}^{2}\|\boldsymbol{W}_i^n(\cdot,t)\|_{\boldsymbol{H}}^2 + \boldsymbol{I}^n(t), \tag{42}$$

where the term $\boldsymbol{I}^n(t)$ represents the convective integrals. With the help of the second energy estimate (26), we estimate $\boldsymbol{I}^n(t)$ as follows:

$$\begin{aligned}
|\boldsymbol{I}^n(t)| &\leq \sum_{i=1}^{2}\int_\Omega |\boldsymbol{W}_i^n(\boldsymbol{x},t)|^2|\nabla_x \boldsymbol{v}_{1,i}^n(\boldsymbol{x},t)|\,d\boldsymbol{x} \\
&\quad + \sum_{i=1}^{2}\int_\Omega |\nabla_x \boldsymbol{W}_i^n(\boldsymbol{x},t)|\,|\boldsymbol{W}_i^n(\boldsymbol{x},t)|\,|\boldsymbol{v}_{2,i}^n(\boldsymbol{x},t)|\,d\boldsymbol{x} \\
&\leq \sum_{i=1}^{2}\|\boldsymbol{W}_i^n(\cdot,t)\|_{L^{2d/(d-2)}(\Omega)^d}^2\|\nabla_x \boldsymbol{v}_{1,i}^n(\cdot,t)\|_{L^{d/2}(\Omega)^{d\times d}} \\
&+ \sum_{i=1}^{2}\|\nabla_x \boldsymbol{W}_i^n(\cdot,t)\|_{L^2(\Omega)^{d\times d}}\|\boldsymbol{W}_i^n(\cdot,t)\|_{L^{2d/(d-2)}(\Omega)^d}\|\boldsymbol{v}_{2,i}^n(\cdot,t)\|_{L^d(\Omega)^d} \\
&\leq M_9\sum_{i=1}^{2}\|\nabla_x \boldsymbol{W}_i^n(\cdot,t)\|_{L^2(\Omega)^{d\times d}}^2\big(\|\Delta_x \boldsymbol{v}_{1,i}^n(\cdot,t)\|_{\boldsymbol{H}} + \|\Delta_x \boldsymbol{v}_{2,i}^n(\cdot,t)\|_{\boldsymbol{H}}\big) \\
&\leq M_{10}\sum_{i=1}^{2}\|\nabla_x \boldsymbol{W}_i^n(\cdot,t)\|_{L^2(\Omega)^{d\times d}}^2,
\end{aligned} \tag{43}$$

where M_9 and M_{10} are positive constants.

Introducing the notation

$$Y(t) := \sum_{i=1}^{2}\big(\|\boldsymbol{W}_i^n(\cdot,t)\|_{\boldsymbol{H}}^2 + \varkappa_i\|\nabla_x \boldsymbol{W}_i^n(\cdot,t)\|_{L^2(\Omega)^{d\times d}}^2\big)$$

and joining (42) and (43), after simple manipulations we arrive at the differential inequality

$$\frac{dY(t)}{dt} \leq \big(M_{10} + |\gamma|\varphi^n(t)\big)Y(t),$$

where $Y(0) = 0$. Using the Gronwall lemma, from this inequality we deduce that $Y(t) \equiv 0$, which amounts to the uniqueness of solution.

Theorem 1 is fully proved.

4 Proof of Theorem 2

4.1. Limiting passage as $n \to +\infty$ in the sequence $\big\{(\boldsymbol{v}_1^n, \boldsymbol{v}_2^n)\big\}$. Proof of assertion 1 of Theorem 2. Due to the uniform in n estimates (11)–(13), assertion 1 of Theorem 2 follows immediately from the Aubin–Lions–Simon compactness theorem [47, Corol. 4] and the Alaoglu theorem.

4.2. Shift and rescaling in the sequence $\{(\boldsymbol{v}_1^n, \boldsymbol{v}_2^n)\}_{n=1,2,\ldots}$. Prior to passing to justification of the limit as $n \to +\infty$ in equations $(9)_{i=1,2}$, we need several preliminary considerations. Assume that the test vector-functions in equations $(9)_{i=1,2}$ belong to $H^1(0,T;\boldsymbol{V})$ and vanish in a neighborhood of the plane $\{t = T\}$. We integrate by parts the first term and the third term in $(9)_{i=1,2}$ in t and write out the resulting equality in an expanded form separating the integrals over segments $(0, 1/n)$ and $(1/n, T)$ from each other:

$$\int_0^{1/n}\int_\Omega \Big(-\boldsymbol{v}_i^n \cdot \partial_t \boldsymbol{\phi}_i + \operatorname{div}_x(\boldsymbol{v}_i^n \otimes \boldsymbol{v}_i^n) \cdot \boldsymbol{\phi}_i - \varkappa_i \nabla_x \boldsymbol{v}_i^n : \nabla_x \partial_t \boldsymbol{\phi}_i + \sum_{j=1}^{2} \mu_{ij} \nabla_x \boldsymbol{v}_j^n : \nabla_x \boldsymbol{\phi}_i - \gamma n \Phi(nt) \boldsymbol{v}_i^n \cdot \boldsymbol{\phi}_i\Big)\, d\boldsymbol{x}dt$$
$$- \int_\Omega \boldsymbol{v}_{0i}(\boldsymbol{x}) \cdot \boldsymbol{\phi}_i(\boldsymbol{x}, 0) d\boldsymbol{x} - \varkappa_i \int_\Omega \nabla_x \boldsymbol{v}_{0i}(\boldsymbol{x}) \cdot \nabla_x \boldsymbol{\phi}_i(\boldsymbol{x}, 0) d\boldsymbol{x}$$
$$+ \int_{1/n}^{T}\int_\Omega \Big(-\boldsymbol{v}_i^n \cdot \partial_t \boldsymbol{\phi}_i + \operatorname{div}_x\big(\boldsymbol{v}_i^n \otimes \boldsymbol{v}_i^n\big) \cdot \boldsymbol{\phi}_i - \varkappa_i \nabla_x \boldsymbol{v}_i^n : \nabla_x \partial_t \boldsymbol{\phi}_i + \sum_{j=1}^{2} \mu_{ij} \nabla_x \boldsymbol{v}_j^n : \nabla_x \boldsymbol{\phi}_i\Big) d\boldsymbol{x}dt = 0 \quad (i = 1, 2), \tag{44}$$

where the fact that the support of the function $t \mapsto \Phi(nt)$ lays in $[0, 1/n]$ is taken into account. In (44), we change the independent variable t and the sought variables $\boldsymbol{v}_i^n$ $(i = 1, 2)$ on the segments $\{0 < t < 1/n\}$ and $\{1/n < t \leq T\}$ as follows. On $(1/n, T]$ we shift the timescale backwards and take

$$\widetilde{t} := t - 1/n, \quad \widetilde{\boldsymbol{v}}_i^n(\boldsymbol{x}, \widetilde{t}) := \boldsymbol{v}_i^n(\boldsymbol{x}, t) \equiv \boldsymbol{v}_i^n(\boldsymbol{x}, \widetilde{t} + 1/n) \quad \text{for } t \in (1/n, T] \quad (i = 1, 2). \tag{45}$$

Note that $\widetilde{t} \in (0, T - 1/n]$, $dt = d\widetilde{t}$, $\partial_t = \partial_{\widetilde{t}}$, and $t = \widetilde{t} + 1/n$. Further, following the idea of rescaling from [50], we take

$$\vartheta := nt, \quad \overline{\boldsymbol{v}}_i^n(\boldsymbol{x}, \vartheta) := \boldsymbol{v}_i^n(\boldsymbol{x}, t) \equiv \boldsymbol{v}_i^n(\boldsymbol{x}, n^{-1}\vartheta) \quad \text{for } t \in [0, 1/n] \quad (i = 1, 2). \tag{46}$$

Note that $\vartheta \in [0, 1]$, $dt = n^{-1}\, d\vartheta$, $\partial_t = n\partial_\vartheta$, and $t = n^{-1}\vartheta$. Thus, (44) takes the form

$$\int_0^1\int_\Omega \Big(-\overline{\boldsymbol{v}}_i^n(\boldsymbol{x}, \vartheta) \cdot \partial_\vartheta \boldsymbol{\phi}_i(\boldsymbol{x}, n^{-1}\vartheta) - \varkappa_i \nabla_x \overline{\boldsymbol{v}}_i^n(\boldsymbol{x}, \vartheta) : \nabla_x \partial_\vartheta \boldsymbol{\phi}_i(\boldsymbol{x}, n^{-1}\vartheta)$$
$$+ n^{-1} \sum_{j=1}^{2} \mu_{ij} \nabla_x \overline{\boldsymbol{v}}_j^n(\boldsymbol{x}, \vartheta) : \nabla_x \boldsymbol{\phi}_i(\boldsymbol{x}, n^{-1}\vartheta)$$

$$
\begin{aligned}
&+ n^{-1}\operatorname{div}_x\big(\overline{\boldsymbol{v}}_i^n(\boldsymbol{x},\vartheta)\otimes\overline{\boldsymbol{v}}_i^n(\boldsymbol{x},\vartheta)\big)\cdot\boldsymbol{\phi}_i(\boldsymbol{x},n^{-1}\vartheta)\\
&\qquad\qquad - \gamma\Phi(\vartheta)\overline{\boldsymbol{v}}_i^n(\boldsymbol{x},\vartheta)\cdot\boldsymbol{\phi}_i(\boldsymbol{x},n^{-1}\vartheta)\Big)d\boldsymbol{x}d\vartheta\\
&- \int_\Omega \boldsymbol{v}_{0i}(\boldsymbol{x})\cdot\boldsymbol{\phi}_i(\boldsymbol{x},0)d\boldsymbol{x} - \varkappa_i\int_\Omega \nabla_x\overline{\boldsymbol{v}}_{0i}(\boldsymbol{x}):\nabla_x\boldsymbol{\phi}_i(\boldsymbol{x},0)d\boldsymbol{x}\\
&+\int_0^{T-1/n}\int_\Omega\Big(-\widetilde{\boldsymbol{v}}_i^n(\boldsymbol{x},\widetilde{t})\cdot\partial_{\widetilde{t}}\boldsymbol{\phi}_i(\boldsymbol{x},\widetilde{t}+1/n) - \varkappa_i\nabla_x\widetilde{\boldsymbol{v}}_i^n(\boldsymbol{x},\widetilde{t})\cdot\nabla_x\partial_{\widetilde{t}}\boldsymbol{\phi}_i(\boldsymbol{x},\widetilde{t}+1/n)\\
&\qquad + \operatorname{div}_x\big(\widetilde{\boldsymbol{v}}_i^n(\boldsymbol{x},\widetilde{t})\otimes\widetilde{\boldsymbol{v}}_i^n(x,\widetilde{t})\big)\cdot\boldsymbol{\phi}_i(\boldsymbol{x},\widetilde{t}+1/n)\\
&\qquad + \sum_{j=1}^2\mu_{ij}\nabla_x\widetilde{\boldsymbol{v}}_j^n(\boldsymbol{x},\widetilde{t}):\nabla_x\boldsymbol{\phi}_i(\boldsymbol{x},\widetilde{t}+1/n)\Big)d\boldsymbol{x}d\widetilde{t} = 0 \quad (i=1,2).
\end{aligned}
\tag{47}
$$

Keeping in mind the further limiting passage as $n\to+\infty$, in (47) we take the test vector-functions $\boldsymbol{\phi}_i = \boldsymbol{\phi}_i^n(\boldsymbol{x},t)$ $(i=1,2)$ depending on n in the following form:

$$
\boldsymbol{\phi}_i^n(\boldsymbol{x},t) = \begin{cases}
\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},\vartheta)\equiv\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},nt) & \text{for } t\in[0,1/n],\\
 & \text{i.e., for } \vartheta\in[0,1],\\
\widetilde{\boldsymbol{\phi}}_i(\boldsymbol{x},\widetilde{t})\equiv\widetilde{\boldsymbol{\phi}}_i(\boldsymbol{x},t-1/n) & \text{for } t\in(1/n,T],\\
 & \text{i.e., for } \widetilde{t}\in(0,T-1/n]
\end{cases}
\quad (i=1,2),
\tag{48}
$$

where $\overline{\boldsymbol{\phi}}_i = \overline{\boldsymbol{\phi}}_i(\boldsymbol{x},\vartheta)$ and $\widetilde{\boldsymbol{\phi}}_i = \widetilde{\boldsymbol{\phi}}_i(\boldsymbol{x},\widetilde{t})$ $(i=1,2)$ are arbitrary smooth test vector-functions defined on $\overline{\Omega}\times[0,1]$ and $\overline{\Omega}\times(0,T]$, respectively, such that $\overline{\boldsymbol{\phi}}_i = \widetilde{\boldsymbol{\phi}}_i\equiv 0$ in a neighborhood of $\partial\Omega$, $\widetilde{\boldsymbol{\phi}}_i\equiv 0$ in a neighborhood of the plane $\{\widetilde{t}=T\}$, and the matching conditions

$$
\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},1-0) = \widetilde{\boldsymbol{\phi}}_i(\boldsymbol{x},0+) \quad (i=1,2)
\tag{49}
$$

hold. Notice that conditions (49) yield that the weak derivatives $\partial_t\boldsymbol{\phi}_i^n$ and $\nabla_x\partial_t\boldsymbol{\phi}_i^n$ $(i=1,2)$ are essentially bounded in Q_T, which implies that $\boldsymbol{\phi}_i^n\in L^2(0,T;\boldsymbol{H})$ and $\partial_t\boldsymbol{\phi}_i^n\in L^2(0,T;\boldsymbol{H})$ $(i=1,2)$. Therefore, $\boldsymbol{\phi}_1^n$ and $\boldsymbol{\phi}_2^n$ are admissible test vector-functions for the respective equations $(44)_{i=1}$ and $(44)_{i=2}$, and, equivalently, for the respective equations $(47)_{i=1}$ and $(47)_{i=2}$. Inserting (48) into (47) (for the respective $i=1$ and $i=2$), we get

$$
\begin{aligned}
&\int_0^1\int_\Omega\Big(-\overline{\boldsymbol{v}}_i^n(\boldsymbol{x},\vartheta)\cdot\partial_\vartheta\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},\vartheta) - \varkappa_i\nabla_x\overline{\boldsymbol{v}}_i^n(\boldsymbol{x},\vartheta):\nabla_x\partial_\vartheta\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},\vartheta)\\
&\qquad + n^{-1}\sum_{j=1}^2\mu_{ij}\nabla_x\overline{\boldsymbol{v}}_j^n(\boldsymbol{x},\vartheta):\nabla_x\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},\vartheta)\\
&\qquad + n^{-1}\operatorname{div}_x\big(\overline{\boldsymbol{v}}_i^n(\boldsymbol{x},\vartheta)\otimes\overline{\boldsymbol{v}}_i^n(\boldsymbol{x},\vartheta)\big)\cdot\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},\vartheta)
\end{aligned}
$$

$$- \gamma\Phi(\vartheta)\overline{\boldsymbol{v}}_i^n(\boldsymbol{x},\vartheta)\cdot\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},\vartheta)\Big)d\boldsymbol{x}d\vartheta$$

$$- \int\limits_\Omega \boldsymbol{v}_{0i}(\boldsymbol{x})\cdot\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},0)d\boldsymbol{x} - \varkappa_i\int\limits_\Omega \nabla_x\overline{\boldsymbol{v}}_{0i}(\boldsymbol{x}) : \nabla_x\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},0)d\boldsymbol{x}$$

$$+ \int\limits_0^{T-1/n}\int\limits_\Omega\Big(-\widetilde{\boldsymbol{v}}_i^n(\boldsymbol{x},\widetilde{t})\cdot\partial_{\widetilde{t}}\widetilde{\boldsymbol{\phi}}_i(\boldsymbol{x},\widetilde{t}) - \varkappa_i\nabla_x\widetilde{\boldsymbol{v}}_i^n(\boldsymbol{x},\widetilde{t})\cdot\nabla_x\partial_{\widetilde{t}}\widetilde{\boldsymbol{\phi}}_i(\boldsymbol{x},\widetilde{t})$$
$$+ \operatorname{div}_x\big(\widetilde{\boldsymbol{v}}_i^n(\boldsymbol{x},\widetilde{t})\otimes\widetilde{\boldsymbol{v}}_i^n(x,\widetilde{t})\big)\cdot\widetilde{\boldsymbol{\phi}}_i(\boldsymbol{x},\widetilde{t})$$
$$+ \sum_{j=1}^2\mu_{ij}\nabla_x\widetilde{\boldsymbol{v}}_j^n(\boldsymbol{x},\widetilde{t}) : \nabla_x\widetilde{\boldsymbol{\phi}}_i(\boldsymbol{x},\widetilde{t})\Big)d\boldsymbol{x}d\widetilde{t} = 0 \quad (i=1,2). \tag{50}$$

Furthermore, we notice that

$$\overline{\boldsymbol{v}}_i^n(\boldsymbol{x},1-0) = \widetilde{\boldsymbol{v}}_i^n(\boldsymbol{x},0+) \quad \text{in } \overline{\Omega} \quad (i=1,2) \tag{51}$$

due to (45), (46) and the regularity properties of $\boldsymbol{v}_i^n$ $(i=1,2)$, see Definition 1.

The rest of the proof of Theorem 1 is based on the systematical study of (50) with account of (51).

4.3. Limiting passage as $n\to+\infty$ in the sequence $\{(\overline{\boldsymbol{v}}_1^n,\overline{\boldsymbol{v}}_2^n)\}$. Proof of assertion 2 of Theorem 2. Applying shift and rescaling (i.e. transformations (45) and (46)) in estimates (11) and (12) and discarding the nonnegative expressions containing $\widetilde{\boldsymbol{v}}_i^n$ $(i=1,2)$, we derive the following uniform estimates for the family $\{(\overline{\boldsymbol{v}}_1^n,\overline{\boldsymbol{v}}_2^n)\}_{n\geq n_0}$:

$$\sum_{i=1}^2\|\overline{\boldsymbol{v}}_i^n\|_{L^\infty(0,1;\boldsymbol{V}^2)} \leq \sum_{i=1}^2\|\boldsymbol{v}_i^n\|_{L^\infty(0,T;\boldsymbol{V}^2)} \leq C_0, \tag{52}$$

$$\sum_{i=1}^2\big(\|\partial_\vartheta\overline{\boldsymbol{v}}_i^n\|_{L^1(0,1;\boldsymbol{H})} + \|\nabla_x\partial_\vartheta\overline{\boldsymbol{v}}_i^n\|_{L^1(0,1;L^2(\Omega)^{d\times d})}\big)$$
$$\leq \sum_{i=1}^2\big(\|\partial_t\boldsymbol{v}_i^n\|_{L^1(0,T;\boldsymbol{H})} + \|\nabla_x\partial_t\boldsymbol{v}_i^n\|_{L^1(0,T;L^2(\Omega)^{d\times d})}\big) \leq C_0, \tag{53}$$

where C_0 is the same constant, as in estimates (11) and (12).

Due to these estimates, assertion 2 of Theorem 2 follows immediately from the Aubin–Lions –Simon compactness theorem and the Alaoglu theorem.

4.4. Limiting passage in $\Omega\times\{0<\vartheta<1\}$. The initial layer equation. Taking $\widetilde{\boldsymbol{\phi}}_i\equiv 0$ in (50), we get

$$\int\limits_0^1\int\limits_\Omega\Big(-\overline{\boldsymbol{v}}_i^n(\boldsymbol{x},\vartheta)\cdot\partial_\vartheta\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},\vartheta) - \varkappa_i\nabla_x\overline{\boldsymbol{v}}_i^n(\boldsymbol{x},\vartheta) : \nabla_x\partial_\vartheta\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},\vartheta)$$

$$
\begin{aligned}
&+ n^{-1}\sum_{j=1}^{2}\mu_{ij}\nabla_x\overline{\boldsymbol{v}}_j^n(\boldsymbol{x},\vartheta):\nabla_x\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},\vartheta)\\
&+ n^{-1}\,\mathrm{div}_x\big(\overline{\boldsymbol{v}}_i^n(\boldsymbol{x},\vartheta)\otimes\overline{\boldsymbol{v}}_i^n(\boldsymbol{x},\vartheta)\big)\cdot\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},\vartheta)\\
&\qquad\qquad\qquad - \gamma\Phi(\vartheta)\overline{\boldsymbol{v}}_i^n(\boldsymbol{x},\vartheta)\cdot\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},\vartheta)\Big)d\boldsymbol{x}d\vartheta\\
&-\int\limits_{\Omega}\boldsymbol{v}_{0i}(\boldsymbol{x})\cdot\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},0)d\boldsymbol{x}-\varkappa_i\int\limits_{\Omega}\nabla_x\overline{\boldsymbol{v}}_{0i}(\boldsymbol{x}):\nabla_x\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},0)d\boldsymbol{x}=0\quad(i=1,2)
\end{aligned}
\tag{54}
$$

for all test vector-functions $\overline{\boldsymbol{\phi}}_i$ $(i=1,2)$ satisfying the conditions imposed above for (48) and vanishing in a neighborhood of the plane $\{\vartheta=1\}$.

Due to the integration by parts formula, elementary inequality $AB\le(A^2+B^2)/2$ $(\forall\,A,B\in\mathbb{R})$, and estimate (52), we have

$$
\begin{aligned}
&\left|\int\limits_0^1\int\limits_{\Omega} n^{-1}\,\mathrm{div}_x\big(\overline{\boldsymbol{v}}_i^n(\boldsymbol{x},\vartheta)\otimes\overline{\boldsymbol{v}}_i^n(\boldsymbol{x},\vartheta)\big)\cdot\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},\vartheta)d\boldsymbol{x}d\vartheta\right|\\
&=\left|\int_0^1\int_{\Omega} n^{-1}(\overline{\boldsymbol{v}}_i^n(\boldsymbol{x},\vartheta)\otimes\overline{\boldsymbol{v}}_i^n(\boldsymbol{x},\vartheta)):\nabla_x\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},\vartheta)\,d\boldsymbol{x}d\vartheta\right|\\
&\le\frac{d}{2n}\|\overline{\boldsymbol{v}}_i^n\|^2_{L^\infty(0,1;\boldsymbol{H}(\Omega))}\|\nabla_x\overline{\boldsymbol{\phi}}_i\|_{C(\overline{\Omega}\times[0,T])}\le\frac{d}{2n}\,C_0^2\|\nabla_x\overline{\boldsymbol{\phi}}_i\|_{C(\overline{\Omega}\times[0,T])}\underset{n\to+\infty}{\longrightarrow}0
\end{aligned}
$$

for $i=1,2$, i.e., the convective terms in (54) tend to zero as $n\to+\infty$. The rest of the integrals in (54) are linear in $\overline{\boldsymbol{v}}_i^n$. The limiting passage in these integrals is based on the limiting relations (16) and causes no difficulties. Thus, as $n\to+\infty$, from (54) we deduce the integral equalities

$$
\begin{aligned}
&\int_0^1\int_{\Omega}\big(-\overline{\boldsymbol{v}}_i(\boldsymbol{x},\vartheta)\cdot\partial_\vartheta\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},\vartheta)-\varkappa_i\nabla_x\overline{\boldsymbol{v}}_i(\boldsymbol{x},\vartheta):\nabla_x\partial_\vartheta\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},\vartheta)\\
&\qquad\qquad\qquad-\gamma\Phi(\vartheta)\overline{\boldsymbol{v}}_i(x,\vartheta)\cdot\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},\vartheta)\big)\,d\boldsymbol{x}d\vartheta\\
&-\int_{\Omega}\boldsymbol{v}_{0i}(\boldsymbol{x})\cdot\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},0)\,d\boldsymbol{x}-\varkappa_i\int_{\Omega}\nabla_x\boldsymbol{v}_{0i}(\boldsymbol{x}):\nabla_x\overline{\boldsymbol{\phi}}_i(\boldsymbol{x},0)\,d\boldsymbol{x}=0\quad(i=1,2)
\end{aligned}
\tag{55}
$$

for all test vector-functions $\overline{\boldsymbol{\phi}}_i$ satisfying the above imposed conditions.

Remark that the integral equalities $(55)_{i=1}$ and $(55)_{i=2}$ are not coupled, since the terms containing coefficients μ_{ij} do not appear in them. Finally, note that the integral equality $(55)_i$ (for each $i=1,2$) is linear in $\overline{\boldsymbol{v}}_i$ and that $(55)_i$ has already been well studied by the classical methods of the theory of generalized solutions of mathematical physics. More certainly, by Theorem 2.1 from [45] we readily conclude that $(55)_i$ along with inclusion $\overline{\boldsymbol{v}}_i\in L^\infty(0,1;\boldsymbol{V}^2)$ is equivalent to system $(17)_i$. In this system, the pressure gradient $\nabla_x\overline{\pi}_i\in L^2(\Omega\times(0,1))^d$ is standardly recovered after the already found solenoidal velocity field $\overline{\boldsymbol{v}}_i$ and the pair $(\overline{\boldsymbol{v}}_i, \overline{\pi}_i)$ is the strong solution of $(55)_i$ in the sense of the first part of Definition 2.

4.5. Limiting passage in $\boldsymbol{\Omega} \times \{0 < \widetilde{t} < T\}$. Equations of the outer flow. Taking $\overline{\boldsymbol{\phi}} \equiv 0$ in (50), we get

$$\int\limits_0^T \int\limits_\Omega \mathbf{1}_{(0<\widetilde{t}<T-1/n)} \Big(-\widetilde{\boldsymbol{v}}_i^n(\boldsymbol{x},\widetilde{t}) \cdot \partial_{\widetilde{t}} \widetilde{\boldsymbol{\phi}}_i(\boldsymbol{x},\widetilde{t}) - \varkappa_i \nabla_x \widetilde{\boldsymbol{v}}_i^n(\boldsymbol{x},\widetilde{t}) \cdot \nabla_x \partial_{\widetilde{t}} \widetilde{\boldsymbol{\phi}}_i(\boldsymbol{x},\widetilde{t}) + \operatorname{div}_x \big(\widetilde{\boldsymbol{v}}_i^n(\boldsymbol{x},\widetilde{t}) \otimes \widetilde{\boldsymbol{v}}_i^n(x,\widetilde{t}) \big) \cdot \widetilde{\boldsymbol{\phi}}_i(\boldsymbol{x},\widetilde{t}) + \sum_{j=1}^2 \mu_{ij} \nabla_x \widetilde{\boldsymbol{v}}_j^n(\boldsymbol{x},\widetilde{t}) : \nabla_x \widetilde{\boldsymbol{\phi}}_i(\boldsymbol{x},\widetilde{t}) \Big) d\boldsymbol{x} d\widetilde{t} = 0 \quad (i = 1,2) \tag{56}$$

for all test vector-functions $\widetilde{\boldsymbol{\phi}}_i$ $(i = 1,2)$ satisfying the conditions imposed above for (48) and vanishing in a neighborhood of the plane $\{\widetilde{t} = 0\}$.

Similarly to estimates (52) and (53) for $\overline{\boldsymbol{v}}_i^n$, using shift and rescaling we derive the following uniform in n estimates for the family $\big\{(\widetilde{\boldsymbol{v}}_1^n, \widetilde{\boldsymbol{v}}_2^n)\big\}_{n \geq n_0}$:

$$\sum_{i=1}^2 \|\widetilde{\boldsymbol{v}}_i^n\|_{L^\infty(0,T-1/n;\boldsymbol{V}^2)} \leq C_0, \tag{57}$$

$$\sum_{i=1}^2 \big(\|\partial_{\widetilde{t}} \widetilde{\boldsymbol{v}}_i^n\|_{L^1(0,T-1/n;\boldsymbol{H})} + \|\nabla_x \partial_{\widetilde{t}} \widetilde{\boldsymbol{v}}_i^n\|_{L^1(0,T-1/n;L^2(\Omega)^{d\times d})} \big) \leq C_0, \tag{58}$$

where C_0 is the same constant, as in estimates (11), (12), (52), and (53). Due to (57), (58), the Aubin–Lions–Simon compactness theorem, and the limiting relation

$$\mathbf{1}_{(0<\widetilde{t}<T-1/n)} \underset{n\to+\infty}{\longrightarrow} 1 \quad \text{strongly in } L^r(0,T) \ \ \forall\, r \in [1,+\infty), \tag{59}$$

there exist a subsequence from $\big\{(\widetilde{\boldsymbol{v}}_1^n, \widetilde{\boldsymbol{v}}_2^n)\big\}$ and a limit pair of vector-functions $(\widetilde{\boldsymbol{v}}_1^n, \widetilde{\boldsymbol{v}}_2^n) \in L^\infty(0,T;\boldsymbol{V}^2)^2$ such that

$$\mathbf{1}_{(0<\widetilde{t}<T-1/n)} \widetilde{\boldsymbol{v}}_i^n \underset{n\to+\infty}{\longrightarrow} \widetilde{\boldsymbol{v}}_i \quad \begin{array}{l} \text{strongly in } L^{2-\nu}(0,T;\boldsymbol{V}), \\ \text{weakly in } L^r(0,T;\, \boldsymbol{V}), \end{array} \quad \forall\, \nu \in (0,1], \quad \forall\, r \in [1,+\infty), \tag{60}$$

for $i = 1,2$.

Recall that, by the Sobolev embedding theorem, $\boldsymbol{V}$ is compactly embedded in $L^4(\Omega)^d$ for $d = 2$ and $d = 3$. From this and relation (60) it follows that

$$\mathbf{1}_{(0<\widetilde{t}<T-1/n)} \widetilde{\boldsymbol{v}}_i^n \otimes \widetilde{\boldsymbol{v}}_i^n \underset{n\to+\infty}{\longrightarrow} \widetilde{\boldsymbol{v}}_i \otimes \widetilde{\boldsymbol{v}}_i \quad \text{strongly in } L^{2-\nu}(0,T;L^2(\Omega)^{d\times d}) \quad (i = 1,2). \tag{61}$$

Further, due to representations (45), estimate (12), and the elementary finite increment formula (see, for example, in [47, § 5, Lem. 4]), we have

$$\int_0^{T-1/n} \|\widetilde{\boldsymbol{v}}_i^n(\cdot,\widetilde{t}) - \boldsymbol{v}_i^n(\cdot,\widetilde{t})\|_{\boldsymbol{H}} d\widetilde{t} \stackrel{(45)}{=} \int_0^{T-1/n} \|\boldsymbol{v}_i^n(\cdot,\widetilde{t}+1/n) - \boldsymbol{v}_i^n(\cdot,\widetilde{t})\|_{\boldsymbol{H}} d\widetilde{t}$$
$$\leq n^{-1} \int_0^T \|\partial_{\widetilde{t}} \boldsymbol{v}_i^n(\cdot,\widetilde{t})\|_{\boldsymbol{H}(\Omega)} d\widetilde{t} \stackrel{(12)}{\leq} n^{-1} C_0 \underset{n\to+\infty}{\longrightarrow} 0 \quad (i=1,2).$$

From this and from the limiting relations (14) and (61) it follows that

$$\widetilde{\boldsymbol{v}}_i(\boldsymbol{x},\widetilde{t}) = \boldsymbol{v}_i(\boldsymbol{x},\widetilde{t}) \quad \text{for a.e. } (\boldsymbol{x},\widetilde{t}) \in Q_T \quad (i=1,2). \tag{62}$$

Now, using relations (59)–(62) and re-denoting $t := \widetilde{t}$, we pass to the limit in (56) as $n \to +\infty$ and by this derive the integral equalities

$$\int_0^T \int_\Omega \Big(-\boldsymbol{v}_i(\boldsymbol{x},t) \cdot \partial_t \widetilde{\boldsymbol{\phi}}_i(\boldsymbol{x},t) - \varkappa_i \nabla_x \boldsymbol{v}_i(\boldsymbol{x},t) : \nabla_x \partial_t \widetilde{\boldsymbol{\phi}}(\boldsymbol{x},t)\big)$$
$$+ \operatorname{div}_x(\boldsymbol{v}_i(\boldsymbol{x},t) \otimes \boldsymbol{v}_i(\boldsymbol{x},t)) \cdot \widetilde{\boldsymbol{\phi}}_i(\boldsymbol{x},t)$$
$$+ \sum_{j=1}^2 \mu_{ij} \nabla_x \boldsymbol{v}_j(\boldsymbol{x},t) : \nabla_x \widetilde{\boldsymbol{\phi}}_i(\boldsymbol{x},t) \Big) d\boldsymbol{x} dt = 0 \quad (i=1,2). \tag{63}$$

Due to the sufficient regularity of test vector-functions $\widetilde{\boldsymbol{\phi}}_i$ $(i=1,2)$, the integral equalities (63) along with inclusions $\boldsymbol{v}_i \in L^\infty(0,T;\boldsymbol{V}^2)$ $(i=1,2)$ are equivalent to system $(18)_{1,2,4}$ in the sense of the theory of distributions. Moreover, the pressure gradients $\nabla_x \pi_i \in L^2(Q_T)^d$ are standardly recovered after the already found solenoidal velocity fields $\overline{\boldsymbol{v}}_i$, and the incompressibility conditions $(18)_2$ (for $i=1,2$) hold a.e. in Q_T. Further, analogously to Lemma 4.1 from [39] or Lemma 3.5 from [37], based on (63) we establish that $\partial_t \boldsymbol{v}_i \in L^\infty(0,T;\boldsymbol{H})$, which yields $\boldsymbol{v}_i \in C([0,T];\boldsymbol{H})$ $(i=1,2)$. Thus, in particular, $\boldsymbol{v}_i(\cdot,0+) \in \boldsymbol{H}$ $(i=1,2)$, i.e., vector-functions $\boldsymbol{v}_i$ have the strong right-sided traces from $\boldsymbol{H}$ on the plane $\{t=0\}$. Also, based on (63), inclusions $\partial_t \boldsymbol{v}_i \in L^\infty(0,T;\boldsymbol{H})$ and $\boldsymbol{v} \in L^\infty(0,T;\boldsymbol{V}^2)$, and the notion of Sobolev weak derivative, using the standard arguments we deduce that $\Delta_x \partial_t \boldsymbol{v} \in L^\infty(0,T;\boldsymbol{H})$. Thus equations $(15)_1$ for $i=1$ and $i=2$ hold a.e. in Q_T and the pair of limit vector-functions $(\boldsymbol{v}_1,\boldsymbol{v}_2)$ satisfies all regularity conditions in (20a).

4.6. Matching condition at $\vartheta = 1-0$. Completion of the proof of assertion 3 of Theorem 2. Justification of conditions $(18)_3$ in this section is a natural modification of arguments from [39, §4.5]. At first, note that, by the finite increment formula [47, §5], from (53) it follows that the family of mappings $\overline{\boldsymbol{v}}_i^n$: $[0,1] \mapsto \boldsymbol{H}$ $(i=1,2)$ is equi-continuous. On the other hand, due to estimate (52) the values of functions $\vartheta \mapsto \overline{\boldsymbol{v}}_i^n(\cdot,\vartheta)$ $(i=1,2)$ belong to the interval $\|\overline{\boldsymbol{v}}_i^n(\cdot,\vartheta)\|_{\boldsymbol{V}} \leq C_0$ $(i=1,2)$. By the Rellich theorem, this interval is a compact set in $\boldsymbol{H}$. Therefore, by the Arcel–Ascoli theorem, the set $\{\overline{\boldsymbol{v}}_i^n\}_{n\geq n_0}$ $(i=1,2)$ is relatively compact in $C([0,1];\boldsymbol{H})$. Consequently, there is a subsequence, still denoted by n, such that $\overline{\boldsymbol{v}}_i^n(\cdot,\vartheta) \underset{n\to+\infty}{\longrightarrow} \overline{\boldsymbol{v}}_i(\cdot,\vartheta)$ in $\boldsymbol{H}$ uniformly on $\{0 \le \vartheta \le 1\}$ for $i = 1, 2$. Quite analogously, from (57) and (58) we deduce that $\widetilde{\boldsymbol{v}}_i^n(\cdot,\widetilde{t}) \underset{n\to+\infty}{\longrightarrow} \widetilde{\boldsymbol{v}}_i(\cdot,\widetilde{t})$ strongly in $\boldsymbol{H}$ uniformly on $\{0 \le \widetilde{t} \le T - 1/n_0\}$ for $i = 1, 2$. These two limiting relations, identities (51) and inclusions $\overline{\boldsymbol{v}}_i(\cdot, 1-0) \in \boldsymbol{H}$ and $\widetilde{\boldsymbol{v}}_i(\cdot, 0+) \overset{(62)}{=} \boldsymbol{v}_i(\cdot, 0+) \in \boldsymbol{H}$ imply that the initial conditions $(18)_3$ hold in the sense of the strong trace sense in $\boldsymbol{H}$. Recall that conditions $(18)_3$ are at the same time the matching conditions at $\vartheta = 1 - 0$.

Theorem 2 is fully proved.

4.7. A note on uniqueness of the solution to problem (17)–(18). Theorem 2.5.2 from [51] and the uniqueness assertion of Theorem 1 directly imply that the weak solution and therefore the strong solution of problem (17)–(18) (in the sense of Definition 2) is unique. In this regard, the pressure functions $\overline{\pi}_i$ and π_i ($i = 1, 2$) are defined uniquely up to a constant value. In turn, since the solution is unique, we conclude that the *entire* family $\{(\boldsymbol{v}_1^n, \boldsymbol{v}_2^n)\}_{n\ge n_0}$ of regular weak solutions of problem (1) tends to the quadruple $(\overline{\boldsymbol{v}}_1, \overline{\boldsymbol{v}}_2, \boldsymbol{v}_1, \boldsymbol{v}_2)$ as $n \to +\infty$ in the sense of relations (14) and (16). Hence, there is no need to extract any subsequence from $\{(\boldsymbol{v}_1^n, \boldsymbol{v}_2^n)\}_{n\to+\infty}$.

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Stanislav Nikolaevich Antontsev
Lavrentyev Institute of Hydrodynamics of the Siberian Branch of the Russian Academy of Sciences,
15, pr. Lavrent'eva,
630090, Novosibirsk, Russia
Email address: antontsevsn@mail.ru

Ivan Vladimirovich Kuznetsov
Lavrentyev Institute of Hydrodynamics of the Siberian Branch of the Russian Academy of Sciences,
15, pr. Lavrent'eva,
630090, Novosibirsk, Russia
Email address: kuznetsov_i@hydro.nsc.ru

Dmitry Alexeyevich Prokudin
Lavrentyev Institute of Hydrodynamics of the Siberian Branch of the Russian Academy of Sciences,
15, pr. Lavrent'eva,
630090, Novosibirsk, Russia
Email address: prokudin@hydro.nsc.ru

Sergey Alexandrovich Sazhenkov
Lavrentyev Institute of Hydrodynamics of the Siberian Branch of the Russian Academy of Sciences,
15, pr. Lavrent'eva,
630090, Novosibirsk, Russia
Email address: sazhenkovs@yandex.ru