S©MR

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ
Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru
ISSN 1813-3304

Vol. 21, No. 2, pp. 1426–1439 (2024) УДК 517.587
https://doi.org/10.33048/semi.2024.21.090 MSC 47D60, 34G10, 47D03, 33C45

HERMITE EXPANSIONS OF $C-$REGULARIZED COSINE FUNCTIONS

R. AMEZIANE HASSANI, Y. BAJJOU, A. BLALI, AND A. EL AMRANI

Communicated by E.M. Rudoy

Abstract: The aim of this paper is to approximate the exponentially bounded $C-$regularized cosine function by the Hermite series, recalling the notions and the results used.

Keywords: Hermite functions, Hermite Expansions, $C-$regularized semigroup, $C-$regularized cosine functions.

1 Introduction and preliminaries

The series expansion of Hermite orthogonal polynomials have been an important tool in quantum mechanics and statistical studies, both theoretical and applied. The study of sufficient conditions for the convergence of Hermite series has been the subject of numerous works ; for more details see [8]. In 2015, L. Abadias and P. Miana studied in their article [4], the Hermite expansion of C_0-groups and cosine functions, in this works we will be interested in Hermite expansion of C-regularized cosine function, starting with reminding the notations, concepts and results used.
Throughout this paper E denotes a non-trivial complex Banach space, $\mathfrak{F}(E,F)$ denotes the set of all applications from E to another Banach space F, $B(E)$ denotes the space of bounded linear operators from E into itself, and $L^1_{loc}(E)$ the set of all $f \in \mathfrak{L}(\mathbb{R}, E)$ locally integrable. For a closed linear operator A

Hassani, R.A., Bajjou, Y., Blali, A., El Amrani, A. Hermite Expansions of $C-$regularized cosine Functions.

Received March, 30, 2024, published December, 25, 2024.

on E, $\mathcal{D}(A)$, $R(A)$ and $\rho(A)$ denote its domain, range and resolvent set, respectively. $\mathcal{D}(A)$ equipped with the graph norm $\| x \|_{\mathcal{D}(A)} = \| x \|_E + \| Ax \|_E$ become Banach space . Throughout this paper, $C \in B(E)$ will be an injective operator. The C-resolvent set of A, denoted by $\rho_C(A)$, is defined by $\rho_C(A) := \{\lambda \in \mathbb{C} \mid R(C) \subseteq R(\lambda I - A) \ \ and \ \ \lambda I - A \text{ is injective in } B(E)\}$ and by $R_C(\lambda, A) = (\lambda I - A)^{-1}C \ \ (\lambda \in \rho_C(A))$ the C-resolvent.

Hermite functions and Hermite Expansions on Banach spaces. For all $n \in \mathbb{N}$, the classical Hermite polynomial is defined by Rodrigues formula:

$$(\forall x \in \mathbb{R}) \ \ H_n(x) = e^{x^2}(-1)^n \frac{d^n}{dx^n} e^{-x^2}.$$

H_n is a polynomial with the of degree n, the same parity as n, whose highest monomial degree is $2^n X^n$ and have real coefficients. Furthermore, they verify the following condition of orthogonality:

$$\int_{\mathbb{R}} H_n(x) H_m(x) e^{-x^2} dx = \delta_{n,m} n! 2^n \sqrt{\pi},$$

where $\delta_{n,m}$ is the Kronecker delta. We also have recurrence relations, differential equations and the Muckenhoupt estimates:

$$(\exists c > 0) \ (\forall (n_0, t) \in \mathbb{N} \times \mathbb{R} : t^2 \leq 2(2n_0 + 1)) \ (\forall n \geq n_0) \mid H_n(t) \mid \leq c\Big(\frac{e^{\frac{t^2}{2}} \sqrt{2^n n!}}{n^{\frac{1}{12}}}\Big). \tag{1}$$

For more details on the classical theory of orthogonal polynomials see [21], [13], [3], [4] and [2].
The Hermite functions on $\mathbb{R}$ are defined by:

$$\phi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{\frac{-x^2}{2}} = (-1)^n \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-x^2}. \tag{2}$$

$(\phi_n)_{n \in \mathbb{N}}$ is an orthonormal basis in the Hilbert space $L^2(\mathbb{R})$ and satisfied some recurrence relations, equality and inequality. For more details See [21], [13], [3], [4] and [2].
For $n \in \mathbb{N}$, we denote by φ_n, the function on $\mathbb{R}$ defined by:

$$(\forall t \in \mathbb{R}) \ \ \varphi_n(t) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{\frac{-t^2}{2}} \phi_n(t) = \frac{1}{2^n n! \sqrt{\pi}} e^{-t^2} H_n(t) = \frac{(-1)^n}{2^n n! \sqrt{\pi}} \frac{d^n}{dt^n} e^{-t^2}. \tag{3}$$

φ_n has the same parity as n, satisfies recurrence relations and differential equations, for example:

$$\left(\forall \, (n,m) \in \mathbb{N}^2\right) \ \ \varphi_n^{(m)} = (-1)^m 2^m (n+1)...(n+m) \varphi_{n+m}. \tag{4}$$

And the following useful inequality given by Cauchy-Schwartz inequality:

$$\| \varphi_n \|_1 \leq \frac{1}{\sqrt{2^n n!}}. \tag{5}$$

For more details see [4].
One of the most important properties of $(\varphi_n)_{n \in \mathbb{N}}$ family is that if $f : \mathbb{R} \to E$

be a differentiable function such that $\int_{-\infty}^{+\infty} e^{-t^2} \parallel f(t) \parallel^2 dt < +\infty$, then the series $\sum_{n\in\mathbb{N}} c_n(f)H_n(t)$ converges pointwise to $f(t)$ for each $t \in \mathbb{R}$, where

$$c_n(f) = \int_{-\infty}^{+\infty} \varphi_n(t)f(t)dt.$$

For more details see [13], [19], [4] and [8].

$C-$regularized semigroups and $C-$regularized cosine functions.

C-regularized semigroups. A map $T : [0, +\infty[\to B(E)$ is called C-regularized semigroup or C-semigroup, if

(1) $T(t+s)C = T(t)T(s)$ for all $t, s \in \mathbb{R}^+$.
(2) $T(0) = C$.
(3) $t \mapsto T(t)x$ is continuous on $\mathbb{R}^+$ for every $x \in E$.

Its generator W is defined by

$$\mathcal{D}(W) = \left\{ x \in E : \lim_{s\to 0^+} \frac{T(s)x - Cx}{s} \text{ exists in R(C) } \right\}$$

and

$$(\forall x \in \mathcal{D}(W)) \quad Wx = C^{-1} \lim_{s\to 0} \frac{T(s)x - Cx}{s}.$$

We say that $(T(t))_{t\geq 0}$ is exponentially bounded $C-$regularized semigroup if $(\exists M \geq 0) \quad (\exists \omega \geq 0) :\parallel T(t) \parallel \leq Me^{\omega t}, \ for \ all \ t \geq 0.$
$(T(t))_{t\geq 0}$ is no degenerate if $T(t)x = 0$ for all $t \geq 0$, then $x = 0$. see [6], [18] and [17].
Since we have for all (t, x) fixed in $\mathbb{R}^+ \times \mathcal{D}(W)$:
$\lim_{s\to 0^+} \frac{T(s)T(t)x - CT(t)x}{s} = T(t) \lim_{s\to 0^+} \frac{T(s)x - Cx}{s} =$
$= T(t)CWx = CT(t)Wx \in R(C)$, this means that $T(t)x \in \mathcal{D}(W)$ and $WT(t)x = T(t)Wx$.
The following results can be found in [6] and [5].
Let $(T(t))_{t\in\mathbb{R}^+}$ be a strongly continuous family such that $\parallel T(t) \parallel \leq Me^{\omega t}$ for all $t \geq 0$ and W is closed linear operator, to conclude that $(T(t))_{t\in\mathbb{R}^+}$ is C-regularized semigroup generated by W, it is sufficient that :

$$W = C^{-1}WC, \ (\omega, +\infty) \subset \rho_C(W),$$

$$and \ R_C(\lambda, W)x = \int_0^{+\infty} e^{-\lambda t} T(t)xdt \quad for \quad \lambda > \omega, x \in E.$$

We refer to [25], [5] and [18] for more details.
Let $0 < \alpha \leq \frac{\pi}{2}, \ \Sigma_\alpha = \{\lambda \in \mathbb{C}/\lambda \neq 0 \ \ and \quad | \, arg(\lambda) \, | < \alpha\}$ and let $(T(t))_{t\geq 0}$ be a C-regularized semigroup. Then we say that $(T(t))_{t\geq 0}$ is an analytic C-regularized semigroup of angle α, if there exists an analytic function $\mathbf{T} : \Sigma_\alpha \to B(E)$ which satisfies:

(1) $\mathbf{T}(t) = T(t) \quad for \quad all \quad t > 0.$
(2) $\lim_{z\to 0, \quad z\in\Sigma_\gamma} \mathbf{T}(z)x = 0$ for all $\gamma \in]0, \alpha[$ and $x \in E$.

For more details see [25].

$C-$**regularized cosine functions.** A map $T:\mathbb{R}\to B(E)$ is called C-regularized cosine function or $C-$cosine function, if

(1) $T(t+s)C+T(t-s)C=2T(t)T(s)$ for all $t,s\in\mathbb{R}$.
(2) $T(0)=C$.
(3) $t\to T(t)x$ is continuous on $\mathbb{R}$ for every $x\in E$.

We say that $(T(t))_{t\in\mathbb{R}}$ is exponentially bounded $C-$regularized cosine if

$$(\exists M>0)\quad(\exists\omega\geq 0):\parallel T(t)\parallel\leq Me^{\omega|t|},\quad for\ \ all\ \ t\in\mathbb{R}.$$

The associated sine operator function $S(.)$ is defined by $S(t):=\int_0^t T(s)ds\ \ for\ \ all\ \ t\in\mathbb{R}$. The operator W defined by

$$\mathcal{D}(W)=\left\{x\in E:\lim_{s\to 0}\frac{2(T(s)x-Cx)}{s^2}\text{ exists in R(C) }\right\}$$

and

$$Wx=C^{-1}\lim_{s\to 0}\frac{2(T(s)x-Cx)}{s^2}\ \ for\ \ all\ \ x\in\mathcal{D}(W)$$

is called the generator of $(T(t))_{t\in\mathbb{R}}$. See [12] and [17].
We notice that W is closed linear operator in E, T is odd, $T(t)x\in\mathcal{D}(W)$, $S(t)x\in\mathcal{D}(W)$, $\int_0^t S(s)xds\in\mathcal{D}(W)$ and $W\int_0^t S(s)xds=T(t)x-Cx$ for all $x\in E$ and $t\in\mathbb{R}$, $C^{-1}WC=W$ and if $(T(t))_{t\in\mathbb{R}}$ is exponentially bounded $C-$regularized cosine then the function sine $S:=(S(t))_{t\in\mathbb{R}}$ is also exponentially bounded, so for all $k\in\mathbb{N}$ and $x\in\mathcal{D}(W)$,

$$\lim_{t\to\pm\infty}t^ke^{-t^2}T(t)x=\lim_{t\to\pm\infty}t^ke^{-t^2}S(t)x=\lim_{t\to\pm\infty}t^ke^{-t^2}WS(t)x=0,\qquad(6)$$

because, if $\parallel T(t)\parallel\leq Me^{\omega|t|}$ for all $t\in\mathbb{R}$, then for all $x\in\mathcal{D}(W)$:
$\parallel t^ke^{-t^2}T(t)x\parallel\leq|\ t\ |^k\ e^{-|t|^2}\ \parallel T(t)x\parallel\leq|\ t\ |^k\ e^{\frac{\omega^2}{4}}e^{-(|t|-\frac{\omega}{2})^2}\parallel x\parallel$. See [12, proposition 1.1 and 1.2] for more properties and more details.
Consider in E the well-posed Cauchy problem

$$((ACP(W,u_0,u_1,0)_2)\begin{cases}u''(t)=Wu(t),\quad t\in\mathbb{R}\\u(0)=u_0\\u'(0)=u_1.\end{cases}$$

Where W generates a C-cosine operator function $(T(t))_{t\in\mathbb{R}}$ then

$$u(t)=C^{-1}T(t)u_0+C^{-1}S(t)u_1$$

is the unique solution of the above Cauchy problem for every pair (u_0,u_1) of initial values in $C(\mathcal{D}(A))$. For more details see [12].
A strongly continuous family $(T(t))_{t\in\mathbb{R}^+}$ such that $\parallel T(t)\parallel\leq Me^{\omega t},\forall t\geq 0$ is $C-$regularized cosine with generator the closed linear operator W if and only if the following condition holds:

$$C^{-1}WC=W,\lambda^2\in\rho_C(W)\text{ and }\lambda R_C(\lambda^2,W)=\int_0^{+\infty}e^{-\lambda t}T(t)dt\text{ on }E\text{ for all }\lambda>\omega.$$

For more details see [11], [12] and [23] .
If W is the generator of exponentially bounded and $C-$regularized cosine $(T(t))_{t\in\mathbb{R}}$, then W is the generator of analytic $C-$regularized semigroup of angle $\frac{\pi}{2}$ defined by

$$(\forall z \in \Sigma_{\frac{\pi}{2}}),\ \ T_1(z)x = \int_0^{+\infty} \frac{e^{\frac{-s^2}{4z}}}{\sqrt{\pi z}} T(s)xds\ \ (for\ all\ z \in \Sigma_{\frac{\pi}{2}}\ , Re(z) > 0) \quad (7)$$

whose proof is similar for C_0-cosine operator-valued function. (See [9] and [1]).

2 Main results

Lemma 1. *Let $(T(t))_{t\in\mathbb{R}}$ be an exponentially bounded C-cosine function on a Banach space E with generator $(W, \mathcal{D}(W))$. For all $n \in \mathbb{N}$ and $x \in \mathcal{D}(W)$, we have :*

(1) $\int_{-\infty}^{+\infty} \varphi_{2n+1}(t)T(t)xdt = 0$

(2) $\int_{-\infty}^{+\infty} \varphi_{2n}(t)T(t)xdt = \frac{1}{2^{2n}(2n)!}W^n T_1(\frac{1}{4})x.$

(3) *In the case where* $\sup\limits_{t\in\mathbb{R}} \| T(t) \| < +\infty$, *we have :*

- *(i)* $\| W^n T_1(\frac{1}{4})x \| \leq 2^n \sqrt{(2n)!} \sup\limits_{t\in\mathbb{R}} \| T(t) \| \| x \|.$
- *(ii)* $\| \int_{-\infty}^{+\infty} \varphi_{2n}(t)T(t)xdt \| \leq \frac{c_1\sqrt{(2(n-1))!}}{2^{n+1}\ (2n)!} \| Wx \|$ *with c_1 a positive constant.*

Proof

(1) Let $n \in \mathbb{N}$ and $x \in E$. The function $t \mapsto \varphi_{2n+1}(t)T(t)x$ is continuous and odd, integrable in the sense of Bochner approach, then

$$\int_{-\infty}^{+\infty} \varphi_{2n+1}(t)T(t)xdt = 0$$

.

(2) Let $x \in \mathcal{D}(W)$. For all $n \in \mathbb{N}$, let $I_n(x) = \int_{-\infty}^{+\infty} \frac{d^{2n}}{dt^{2n}}(e^{-t^2})T(t)xdt$, then:

$$\begin{aligned}
I_n(x) &= [\frac{d^{2n-1}}{dt^{2n-1}}(e^{-t^2})T(t)x]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \frac{d^{2n-1}}{dt^{2n-1}}(e^{-t^2}) \frac{d}{dt}T(t)xdt \\
&= 0 - [\frac{d^{2n-2}}{dt^{2n-2}}(e^{-t^2})WS(t)x]_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \frac{d^{2n-2}}{dt^{2n-2}}(e^{-t^2}) \frac{d^2}{dt^2}T(t)xdt \\
&= 0 - 0 + \int_{-\infty}^{+\infty} \frac{d^{2n-2}}{dt^{2n-2}}(e^{-t^2})WT(t)xdt \\
&= W \int_{-\infty}^{+\infty} \frac{d^{2n-2}}{dt^{2n-2}}(e^{-t^2})T(t)xdt \\
&= W\ I_{n-1}(x).
\end{aligned}$$

A simple recurrence on n gives

$$I_n(x) = W^n I_0(x) = W^n \int_{-\infty}^{+\infty} e^{-t^2} T(t) x dt.$$

By definition of φ_n; from where we obtain:

$$\begin{aligned}
\int_{-\infty}^{+\infty} \varphi_{2n}(t) T(t) x dt &= \int_{-\infty}^{+\infty} \frac{(-1)^{2n}}{2^{2n}(2n)!\sqrt{\pi}} \frac{d^{2n}}{dt^{2n}} e^{-t^2} T(t) x dt \\
&= \frac{1}{2^{2n}(2n)!\sqrt{\pi}} I_n(x) \\
&= \frac{W^n}{2^{2n}(2n)!\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2} T(t) x dt \\
&= \frac{W^n}{2^{2n}(2n)!\sqrt{\pi}} 2 \int_{0}^{+\infty} e^{-t^2} T(t) x dt \\
&= \frac{W^n}{2^{2n}(2n)!} T_1(\frac{1}{4}) x.
\end{aligned}$$

(3) In the case where $\sup\limits_{t \in \mathbb{R}} \| T(t) \| < +\infty$, we obtain:

- (i) For all $x \in \mathcal{D}(W)$:

$$\begin{aligned}
\| W^n T_1(\frac{1}{4}) x \| &\leq 2^{2n}(2n)! \int_{-\infty}^{+\infty} \| \varphi_{2n}(t) T(t) x \| dt \\
&\leq 2^{2n}(2n)! \left(\sup_{t \in \mathbb{R}} \| T(t) \| \right) (\| x \|)(\| \varphi_{2n}(t) \|_1) \\
&\leq 2^{2n}(2n)! \sup_{t \in \mathbb{R}} \| T(t) \| \times \| x \| \times \frac{1}{\sqrt{2^{2n}(2n)!}} \quad (by \;\; 5) \\
&\leq 2^n \sqrt{(2n)!} \| x \| \sup_{t \in \mathbb{R}} \| T(t) \| .
\end{aligned}$$

- (ii) Let $x \in \mathcal{D}(W)$ and $n \geq 1$, if we posed $B = \| \int_{-\infty}^{+\infty} \varphi_{2n}(t) T(t) x dt \|$; then using (i) and integration by parts we get

$$
\begin{aligned}
B &= \parallel \frac{1}{4(2n-1)(2n)} \int_{-\infty}^{+\infty} \varphi''_{2n-2}(t)T(t)xdt \parallel \quad (by \;\; 4) \\
&= \parallel -\frac{1}{4(2n-1)(2n)} \int_{-\infty}^{+\infty} \varphi'_{2n-2}(t)T'(t)xdt \parallel \\
&= \parallel \frac{1}{4(2n-1)(2n)} \int_{-\infty}^{+\infty} \varphi_{2n-2}(t)T''(t)xdt \parallel \\
&= \parallel \frac{1}{4(2n-1)(2n)} \int_{-\infty}^{+\infty} \varphi_{2n-2}(t)WT(t)xdt \parallel \;\; (like \;1) \\
&\leq \frac{1}{4(2n-1)(2n)} \left(\int_{-\infty}^{+\infty} \mid \varphi_{2n-2}(t) \mid dt\right) \left(\sup_{t\in\mathbb{R}} \parallel T(t) \parallel\right) (\parallel Wx \parallel) \\
&\leq \left(\sup_{t\in\mathbb{R}} \parallel T(t) \parallel\right) \frac{1}{4(2n-1)(2n)} \parallel \varphi_{2n-2} \parallel_1 \;\; \parallel Wx \parallel \\
&\leq \frac{c_1}{2^2(2n-1)(2n)} \frac{1}{2^{n-1}\sqrt{2(n-1)!}} \parallel Wx \parallel \quad (by \;\; 5) \;\; (c_1 = \sup_{t\in\mathbb{R}} \parallel T(t) \parallel) \\
&\leq \frac{c_1\sqrt{2(n-1)!}}{(2n-1)(2n)} \frac{1}{2^{n+1}(2(n-1))!} \parallel Wx \parallel \\
&\leq \frac{c_1\sqrt{2(n-1)!}}{2^{n+1}(2n)!} \parallel Wx \parallel .
\end{aligned}
$$

Theorem 1. *Let $(T(t))_{t\in\mathbb{R}}$ be an exponentially bounded $C-$cosine function on a Banach space E ($T(t) \leq Me^{\omega|t|}, M > 0$ and $\omega \geq 0$) with a generator $(W, \mathcal{D}(W))$.*

(1) For any $x \in \mathcal{D}(W)$,

$$T(t)x = \sum_{n=0}^{+\infty} \frac{1}{2^{2n}(2n)!} W^n T_1(\frac{1}{4}) x H_{2n}(t), \;\; for \; all \; t \in \mathbb{R}.$$

(2) In the case where $\sup_{t\in\mathbb{R}} \parallel T(t) \parallel < +\infty$, we have for any $x \in \mathcal{D}(W)$:

- *(i)*

$$T_1(z)x = \sum_{n=0}^{+\infty} \frac{1}{2^{2n}n!} W^n T_1(\frac{1}{4}) x (4z-1)^n, \;\; for \; all \; z \in \mathbb{C}, \; \mid z - \frac{1}{4} \mid < \frac{1}{4}.$$

- *(ii) For all $t \in \mathbb{R}$, there is $m_t \in \mathbb{N}$ and $c_t > 0$ such that:*

$$(\forall m \geq m_t) \;\; \parallel T(t)x - \sum_{n=0}^{m} \frac{1}{2^{2n}(2n)!} W^n T_1(\frac{1}{4}) x H_{2n}(t) \parallel \leq \frac{C_t}{m^{\frac{1}{12}}} \parallel Wx \parallel .$$

Moreover, the convergence is uniform on any compact of $\mathbb{R}$.

Proof

(1) Let $x \in \mathcal{D}(W)$, which implies that $Cx \in \mathcal{D}(W)$ because $WCx = CWx$. Let's remember that $T(.)x : \mathbb{R} \to E$ is in $C^2(\mathbb{R}, E)$ and like

$$\begin{aligned}
\int_{-\infty}^{+\infty} e^{-t^2} \parallel T(t)x \parallel^2 dt &\leq \int_{-\infty}^{+\infty} e^{-t^2} \parallel T(t) \parallel^2 \parallel x \parallel^2 dt \\
&\leq M^2 \int_{-\infty}^{+\infty} e^{-t^2} e^{2\omega|t|} \parallel x \parallel^2 dt \\
&\leq M^2 \parallel x \parallel^2 \int_{-\infty}^{+\infty} e^{-|t|^2+2\omega|t|} dt \\
&\leq M^2 \parallel x \parallel^2 e^{\omega^2} \int_{-\infty}^{+\infty} e^{-(|t|-\omega)^2} dt \\
&\leq M^2 \parallel x \parallel^2 e^{\omega^2} \int_{-\infty}^{+\infty} e^{-u^2} du \\
&< +\infty,
\end{aligned}$$

then, the series $\displaystyle\sum_{n\in\mathbb{N}} c_n(T(.)x)H_n$ with

$$c_n(T(.)x) = \textstyle\int_{-\infty}^{+\infty} \varphi_n(t)T(t)xdt = \frac{1}{2^n n!} W^n T_1(\frac{1}{4})x$$

converges pointwise to $T(t)x$ for $t \in \mathbb{R}$, it is $T(t)x = \displaystyle\sum_{n=0}^{+\infty} \frac{1}{2^n n!} W^n T_1(\frac{1}{4}) x H_n(t)$.

(2) • (i)
According to 3.(i) of Lemma 1, for all $x \in \mathcal{D}(W)$ and $z \in \mathbb{C}$, we have

$$\begin{aligned}
\sum_{n=0}^{+\infty} \frac{\parallel W^n T_1(\frac{1}{4})x \parallel}{2^{2n} n!} \mid 4z-1 \mid^n &\leq \sum_{n=0}^{+\infty} \frac{2^n \sqrt{(2n)!}}{2^{2n} n!} \mid 4z-1 \mid^n \parallel x \parallel \left(\sup_{t\in\mathbb{R}} \parallel T(t) \parallel \right) \\
&\leq c_1 \sum_{n=0}^{+\infty} \frac{\sqrt{(2n)!}}{2^n n!} \mid 4z-1 \mid^n,
\end{aligned}$$

where $c_1 = \left(\sup_{t\in\mathbb{R}} \parallel T(t) \parallel \right) \parallel x \parallel$.

Stirling's formula $\frac{\sqrt{(2n)!}}{n!} \sim_{+\infty} 2^n n^{\frac{-1}{4}} (2\pi)^{\frac{-1}{4}} 2^{\frac{1}{4}}$ implies that:

$c_1 \displaystyle\sum_{n=0}^{+\infty} \frac{\sqrt{(2n)!}}{2^n n!} \mid 4z-1 \mid^n \leq c \sum_{n=0}^{+\infty} \frac{\mid 4z-1 \mid^n}{n^{\frac{1}{4}}} < \infty$, for $\mid z - \frac{1}{4} \mid < \frac{1}{4}$.

Finally let a analytic family of operators :

$V : D(\frac{1}{4}, \frac{1}{4}) = \{z \in \mathbb{C} / \mid z - \frac{1}{4} \mid < \frac{1}{4}\} \to B(\mathcal{D}(W)), z \mapsto V(z) :$

$\mathcal{D}(W) \to \mathcal{D}(W),\ x \mapsto V(z)x = \displaystyle\sum_{n=0}^{+\infty} \frac{W^n T_1(\frac{1}{4})x}{2^{2n} n!} (4z-1)^n.$

So for $x \in D(W)$:

$$\begin{aligned} V'(z)x &= \sum_{n=1}^{+\infty} \frac{W^n T_1(\frac{1}{4})x}{2^{2n} n!} 4n(4z-1)^{n-1} \\ &= \sum_{n=0}^{+\infty} \frac{W^n T_1(\frac{1}{4})Wx}{2^{2n} n!} (4z-1)^n \\ &= V(z)Wx \\ &= WV(z)x. \end{aligned}$$

Like W a generator of analytic $C-$semigroups $(T_1(t))_{t\geq 0}$ of angle $\frac{\pi}{2}$ and $V(\frac{1}{4})x = T_1(\frac{1}{4})x \in \mathcal{D}(W)$ then $V(t)x = T_1(t)x$ for $t \in]0, \frac{1}{2}[$ because the first-order abstract Cauchy problem

$$\begin{cases} u'(t) = Wu(t), & t \in]0, \frac{1}{2}[\\ u(\frac{1}{4}) = T_1(\frac{1}{4})x \end{cases}$$

has unique solution defined in $]0, \frac{1}{2}[$. On the other hand the two functions $z \mapsto T_1(z)x$ and $z \mapsto V(z)x$ are holomorphic on the connected open set $D(\frac{1}{4}, \frac{1}{4})$, likewise $]0, \frac{1}{2}[\subset \{u \in D(\frac{1}{4}, \frac{1}{4}),\ V(u)x = T_1(u)x\}$, by the principle of analytical continuation we get: $V(z)x = T_1(z)x$ for $\mid z - \frac{1}{4} \mid < \frac{1}{4}$.

- (ii) Let $t \in \mathbb{R}$ be fixed. We are looking to increase the quantity

$$\| \sum_{n=0}^{+\infty} \frac{1}{2^{2n}(2n)!} W^n T_1(\frac{1}{4}) x H_{2n}(t) - \sum_{n=0}^{m} \frac{1}{2^{2n}(2n)!} W^n T_1(\frac{1}{4}) x H_{2n}(t) \| . \quad (8)$$

There exists $m_t \in \mathbb{N}^*$ such that $0 \leq t^2 \leq 2(2m_t + 1)$, according to inequality (1)

$(\forall n \geq m_t) \ \mid H_{2n}(t) \mid \leq c \left(\frac{e^{\frac{t^2}{2}} \sqrt{2^{2n}(2n)!}}{(2n)^{\frac{1}{12}}} \right)$ (the c is independent of t and it is the inequality constant (1)),

let $x \in \mathcal{D}(W)$, according to the inequality $3 - (ii)$ of lemma 1, there is a constant $c_1 > 0$ such as for all $m \geq m_t$ from where:

$$
\begin{aligned}
(8) \quad & \leq \sum_{n=m+1}^{+\infty} \| \frac{1}{2^{2n}(2n)!} W^n T_1(\frac{1}{4}) x H_{2n}(t) \| \\
& \leq \sum_{n=m+1}^{+\infty} \| \int_{-\infty}^{+\infty} \varphi_{2n}(s) T(s) x ds \| \ \mid H_{2n}(t) \mid \\
& \leq \sum_{n=m+1}^{+\infty} \frac{c_1 \sqrt{(2n-2)!}}{2^{n+1}(2n)!} \| Wx \| \, c \frac{e^{\frac{t^2}{2}} \sqrt{2^{2n}(2n)!}}{(2n)^{\frac{1}{12}}} \quad (3.ii \ of \ lemma \ 1) \\
& \leq \sum_{n=m+1}^{+\infty} \frac{c c_1 e^{2m_t+1} \sqrt{(2n-2)!(2n)!} 2^n}{2^{n+1}(2n)! 2^{\frac{1}{12}} n^{\frac{1}{12}}} \| Wx \| \\
& \leq \sum_{n=m+1}^{+\infty} c_t^{'} \sqrt{\frac{(2n-2)!}{(2n)!}} \frac{1}{n^{\frac{1}{12}}} \| Wx \| \quad (c_t^{'} = c c_1 e^{2m_t+1}) \\
& \leq \sum_{n=m+1}^{+\infty} \frac{c_t^{'}}{\sqrt{2n(2n-1)} n^{\frac{1}{12}}} \| Wx \| \\
& \leq \sum_{n=m+1}^{+\infty} \frac{c_t^{'}}{n^{\frac{13}{12}}} \| Wx \| \quad (because \ \frac{1}{\sqrt{2n(2n-1)}} \leq \frac{1}{n}).
\end{aligned}
$$

The Riemann serie $\sum_{n\geq 1} \frac{1}{n^{\frac{13}{12}}}$ is convergent, so

$$\sum_{n=m+1}^{+\infty} \frac{1}{n^{\frac{13}{12}}} \sim_{m\to+\infty} \frac{1}{\frac{13}{12}-1} \frac{1}{m^{\frac{13}{12}-1}} = \frac{1}{\frac{1}{12}} \frac{1}{m^{\frac{1}{12}}} \text{ from where}$$

$$
\begin{aligned}
\| T(t)x - \sum_{n=0}^{m} \frac{1}{2^{2n}(2n)!} W^n T_1(\frac{1}{4}) x H_{2n}(t) \| \quad & \leq \quad c_t^{'} c_2 \frac{12}{m^{\frac{1}{12}}} \| Wx \| \\
& \leq \quad \frac{c_t}{m^{\frac{1}{12}}} \| Wx \| .
\end{aligned}
$$

- If $K \subset \mathbb{R}$ is a compact set then there is $m_0 \in \mathbb{N}$ for all $t \in K$, $0 \leq t^2 \leq 2(2m_0+1)$, so the constant $c_t^{'}$ (from where c_t) is independent of t therefore, the reminder tends uniformly on K towards 0.

Example 1. *Let $m : \mathbb{R} \to \mathbb{R}^-$ be an even measurable function. In the Banach space $L^1(\mathbb{R})$, we consider the family $T := (T(t))_{t\in\mathbb{R}} \subset \mathfrak{F}(L^1(\mathbb{R}))$ defined by $T : \mathbb{R} \to B\left(L^1(\mathbb{R})\right), t \mapsto T(t) : L^1(\mathbb{R}) \to L^1(\mathbb{R}), f \mapsto T(t)(f) : \mathbb{R} \to \mathbb{R}, s \mapsto T(t)(f)(s) = \cos\left(t\sqrt{-m(s)}\right) f(-s)$.*
Clearly, $(T(t))_{t\in\mathbb{R}} \subset B(L^1(\mathbb{R}))$. If we put $T(0) = C$, then for all $s, t \in \mathbb{R}$ $T(t)(f)(s) = \cos\left(t\sqrt{-m(s)}\right) C(f)(s)$. We also have:

- *(i) C is an injective element of $B\left(L^1(\mathbb{R})\right)$.*

- *(ii) For all $t, s, z \in \mathbb{R}$:*

$$\begin{aligned} D(z) &= (T(t+s)+T(t-s))\,(C(f))\,(z) \\ &= T(t+s)\,(C(f))\,(z)+T(t-s)\,(C(f))\,(z) \\ &= \cos\left((t+s)\sqrt{-m(z)}\right)C(f)(-z)+\cos\left((t-s)\sqrt{-m(z)}\right)C(f)(-z) \\ &= \cos\left((t+s)\sqrt{-m(z)}\right)f(z)+\cos\left((t-s)\sqrt{-m(z)}\right)f(z) \\ &= \left(\cos\left(t\sqrt{-m(z)}+s\sqrt{-m(z)}\right)+\cos\left(t\sqrt{-m(z)}-s\sqrt{-m(z)}\right)\right)f(z) \\ &= 2\cos\left(t\sqrt{-m(z)}\right)\cos\left(s\sqrt{-m(z)}\right)f(z) \\ &= 2\cos\left(t\sqrt{-m(z)}\right)\cos\left(s\sqrt{-m(z)}\right)(C(f))\,(-z) \\ &= 2\cos\left(t\sqrt{-m(z)}\right)T(s)\,(C(f))\,(z) \\ &= 2\cos\left(t\sqrt{-m(z)}\right)C\,(T(s)(f))\,(z) \quad (m \textit{ is an even function}) \\ &= 2T(t)\,(T(s)(f))\,(z) \\ &= 2T(t)T(s)(f)(z). \end{aligned}$$

- *(iii) For each f fixed in $L^1(\mathbb{R})$. The function $\mathbb{R} \to L^1(\mathbb{R}), t \mapsto T(t)(f)$ is continuous on $\mathbb{R}$.*
- *(iv) For all $f, g \in L^1(\mathbb{R})$ such that $C^{-1}\lim_{t\to 0}\dfrac{2(T(t)f-C(f))}{t^2}=g$, we have :*

$$\begin{aligned} 0 &= \lim_{t\to 0}\| \frac{2\left(T(t)(f)-C(f)\right)}{t^2}-C(g)\|_{L^1(\mathbb{R})} \\ &= \lim_{t\to 0}\int_{\mathbb{R}} | \frac{2\left(\cos\left(t\sqrt{-m(s)}\right)C(f)(s)-C(f)(s)\right)}{t^2}-C(g)(s) \mid ds \\ &= \lim_{t\to 0}\int_{\mathbb{R}} | \frac{2\left(\cos\left(t\sqrt{-m(s)}\right)-1\right)}{t^2}C(f)(s)-C(g)(s) \mid ds \\ &= \int_{\mathbb{R}}\lim_{t\to 0} | \frac{2\left(\cos\left(t\sqrt{-m(s)}\right)-1\right)}{t^2}C(f)(s)-C(g)(s) \mid ds \ (Fatou's\ lemma) \\ &= \int_{\mathbb{R}} | m(s).C(f)(s)-C(g)(s) \mid ds \\ &= \int_{\mathbb{R}} | C\left(C(m).f-g\right)(s) \mid ds \ (\textit{ because } \forall\ h \in\ \mathfrak{F}(\mathbb{R},\mathbb{R})\ \ C\circ C(h)=h\ ) \\ &= \| C\left(m.f-g\right)\|_{L^1(\mathbb{R})}, \ \ (C(m)=m \ \textit{ because } \ m \ \textit{ is an even function}). \end{aligned}$$

- *(v) Let $t \in \mathbb{R}$ fixed, we have for all $f \in L^1(\mathbb{R})$:* $\| T(t)(f)\|_{L^1(\mathbb{R})}=\int_{\mathbb{R}} | \cos\left(t\sqrt{-m(s)}\right)f(-s) \mid ds \leq \| f \|_{L^1(\mathbb{R})}$.

Ultimately, $(T(t))_{t\in\mathbb{R}}$ is uniformly bounded $C-$regularized cosine function with generator $(W, \mathcal{D}(W))$ defined by W: $\mathcal{D}(W)=\{f \in L^1(\mathbb{R})/\ m.f \in L^1(\mathbb{R})\} \to L^1(\mathbb{R}),\ f \mapsto W(f)=m.f$. *On the* *other hand, for all* $s \in \mathbb{R}$ *and* $f \in L^1(\mathbb{R})$,

$$\begin{aligned}
\frac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-t^2}T(t)(f)(s)dt &= \frac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-t^2}\cos\left(t\sqrt{-m(s)}\right)f(-s)dt \\
&= \frac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-t^2}Re\left(e^{i\left(t\sqrt{-m(s)}\right)}\right)f(-s)dt \\
&= Re\left(\tfrac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-t^2+it\sqrt{-m(s)}}f(-s)dt\right) \\
&= Re\left(\tfrac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{\frac{m(s)}{4}}e^{-\left(t-i\frac{\sqrt{-m(s)}}{2}\right)^2}f(-s)dt\right) \\
&= e^{\frac{m(s)}{4}}\left(\tfrac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-u^2}du\right).f(-s) \\
&= e^{\frac{m(s)}{4}}f(-s),
\end{aligned}$$

then for all $s \in \mathbb{R}$ *and* $f \in L^1(\mathbb{R})$

$$T_1(\frac{1}{4})(f)(s) = \left(\tfrac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-t^2}\cos(t\sqrt{-m(s)})dt\right).f(-s) = e^{\frac{m(s)}{4}}f(-s),$$

Theorem 1 gives for $f \in \mathcal{D}(W)$ *and* $s, t \in \mathbb{R}$:

$$\begin{aligned}
T(t)(f)(s) &= \sum_{n=0}^{+\infty}\frac{1}{2^{2n}(2n)!}W^nT_1(\frac{1}{4})(f)(s)H_{2n}(t) \\
&= \sum_{n=0}^{+\infty}\frac{(m(s))^n}{2^{2n}(2n)!}e^{\frac{m(s)}{4}}f(-s)H_{2n}(t).
\end{aligned}$$

So $T(t)(f) = \displaystyle\sum_{n=0}^{+\infty}\frac{(m(.))^n}{2^{2n}(2n)!}e^{\frac{m(.)}{4}}H_{2n}(t)C(f).$

Acknowledgments The authors are greatly indebted to the editor and the referee for many valuable comments and suggestions which improved the initial draft of this paper.

References

[1] W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Birkhäuser Verlag, Basel, 2001. Zbl 0978.34001

[2] R. Askey, S. Wainger, *Mean convergence of expansions in Laguerre and Hermite series*, Amer. J. Math., **87**:3 (1965), 695–708. Zbl 0125.31301

[3] S.S. Bonan, D. S. Clark, *Estimates of the Hermite and the Freud Polynomials*, J. Approx. Theory, **63**:2 (1990), 210–224. Zbl 0716.42018

[4] L. Abadias, P. Miana, *Hermite expansions of C_0-groups and cosine functions*, J. Math. Anal. Appl., **426**:1 (2015), 288–311. Zbl 1321.47032

[5] R. Delaubenfels, S. Guozheng, S. Wang, *Regularized semigroups, Existence Familles And the abstract Cauchy Problem*, Differ. Integral Equ., **8**:6 (1995), 1477–1496. Zbl 0823.47033

[6] F.L. Schwenninger, *Generalisations of Semigroups of Operators in the View of Linear Relations*, Institut für Analysis und Scientific Computing der Technischen Universität Wien, 2011.

[7] V. Keyantuo, C. Lizama, P.J. Miana, *Algebra homomorphisms defined via convoluted semigroups and cosine functions*, J. Funct. Anal., **257**:11 (2009), 3454–3487. Zbl 1190.47045

[8] E. Kogbetliantz, *Recherches sur la sommabilite des series d'Hermite*, Ann. Sci. Éc. Norm. Supér., **49**:3 (1932), 137–221. Zbl 0005.15701

[9] M. Kostic, *$(a, k)-$Regularized $C-$Resolvent Families: Regularity and Local Properties*, Abstr. Appl. Anal, **2009**, Article ID 858242, 27 p. Zbl 1200.47059

[10] M. Kostic, *Approximation and convergence of degenerate (a, k)-regularized C-resolvent families*, Bull., Cl. Sci. Math. Nat., Sci. Math., **42** (2017), 69–83. Zbl 1474.45078

[11] C.-C. Kuo, *Perturbations of local $C-$cosine functions*, Stud. Univ. Babeş-Bolyai, Math., **65**:4 (2020), 585–597. Zbl 1504.47070

[12] C.-C. Kuo, S.-Y. Shaw, *$C-$Cosine Functions and the Abstract Cauchy Problem*, J. Math. Anal. Appl., **210**:2 (1997), 632–646. Zbl 0881.34071

[13] N.N. Lebedev, *Special functions and their applications. Translated from the Russian by R. A. Silverman*, Englewood Cliffs, N.J.: Prentice Hall, Inc. XII, 1965. Zbl 0131.07002

[14] F. Li, J. Liang, T.J. Xiao, J. Zhang, *On perturbation of convoluted $C-$regularized operator families*, J. Funct. Spaces Appl., **2013**, Article ID 579326, 8 p. Zbl 1286.47027

[15] F. Li, J.H. Liu, *Note on multiplicative perturbation of local $C-$regularized cosine functions with nondensely defined generators*, Electron. J. Qual. Theory Differ. Equ., **2010**, Paper No. 57, 12p. Zbl 1217.47083

[16] M. Li, Q. Zheng, J. Zhang, *Regularized resolvent families*, Taiwanese J. Math., **11**:1 (2007), 117–133. Zbl 1157.45006s

[17] M. Mosallanezhada, M. Janfada, *On mixed $C-$semigroups of Operators on Banach Spaces*, Filomat, **30**:10 (2016), 2673–2682. Zbl 1465.47035

[18] P. Preda, A. Pogan, C. Preda, *The Perron problem for $C-$semigroups*, Math. J. Okayama Univ., **46** (2004), 141–151. Zbl 1091.47036

[19] J. A. Shohat, *On the development of functions in series of orthogonal polynomials*, Bull. Am. Math. Soc., **41** (1935), 49–82. Zbl 0011.15601

[20] H. Skovgaard, *Asymptotic forms of hermite polynomials*, Department of Mathematics California institute of Technology, Pasadenea, 1959.

[21] G. Szegő, *Asymptotic forms of hermite polynomials*, American Mathematical Society (AMS), Providence, RI, 1939. Zbl 0023.21505

[22] S.W. Wang, *Properties of subgenerators of C-Regularized semigroups*, Proc. Am. Math. Soc., **126**:2 (1998), 453–460. Zbl 0892.47039

[23] J. Zhang, *Regularized cosine Functions and Polynomials of Group Generators*, Sci. Iran., **4**:1–2 (1997), 12–18. Zbl 0973.47032

[24] Q. Zheng, *Matrices of operators and regularized cosine functions*, J. Math. Anal. Appl., **315**:1 (2006), 68–75. Zbl 1092.47039

[25] Q. Zheng, Y. Zhao, *An analyticity criterion for regularized semigroups*, Proc. Am. Math., **130**:8 (2002), 2271–2274. Zbl 1015.47030.

Rachid Ameziane Hassani
Department of mathematics and computer science,
Dhar El mahraz Faculty of Sciences,
Sidi Mohamed Ben Abdellah University,
Fès Atlas, Morocco
Email address: ra.ameziane@yahoo.fr

Youssef Bajjou
Department of mathematics and computer science,
Dhar El mahraz Faculty of Sciences,
Sidi Mohamed Ben Abdellah University,
Fès Atlas, Morocco
Email address: youssefbajjou2017@gmail.com

Aziz Blali
Department of mathematics,
Ecole Normale Supérieure,
Sidi Mohamed Ben Abdellah University,
B.P. 5206 Bensouda-Fes, Morocco
Email address: aziz.blali@usmba.ac.ma

Abdelkhalek El Amrani
Department of mathematics and computer science,
Dhar El mahraz Faculty of Sciences,
Sidi Mohamed Ben Abdellah University,
Fès Atlas, Morocco
Email address: abdelkhalek.elamrani@usmba.ac.ma