ON THE GELFAND PROBLEM AND VISCOSITY MATRICES FOR TWO-DIMENSIONAL HYPERBOLIC SYSTEMS OF CONSERVATION LAWS

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Dedicated to the memory of Sergey Godunov

Abstract: We present counter-intuitive examples of viscous regularizations of a two-dimensional strictly hyperbolic systems of conservation laws. The regularizations are obtained using two different viscosity matrices. While for both of the constructed "viscous" systems waves propagating in either x- or y-directions are stable, oblique waves may be linearly unstable. Numerical simulations fully corroborate these analytical results. To the best of our knowledge, this is the first nontrivial result related to the multidimensional Gelfand problem with non-symmetric fluxes and diffusion terms. Our conjectures provide direct answer to Gelfand's problem both in one- and multi-dimensional cases.

Keywords: Viscosity matrices, hyperbolic systems of conservation laws, Saint-Venant system of shallow water equations.

1 Introduction

Hyperbolic systems of conservation laws and their viscous regularizations pertaining to the natural sciences are among the most important partial differential equations (PDEs) and invariably stir the interest of mathematicians of all walks. Stability of solutions and their dependence on parameters are among the basic and most important questions that need to be investigated. During the height of the Cold War, Soviet-American mathematician I. M. Gelfand published a paper with an unassuming title "Some problems in the theory of quasi-linear equations" [5]. It was the time when high-temperature physics, nuclear reactions, and powerful explosions became the hotly pursued and studied areas as tools of political wrestling. Being inspired by the mathematical problems relevant to the new realms of physics, with his (un)usual insight and tour de force, Gelfand formulated and posted a few fundamental problems in the theory of PDEs.

Among them, one of the most intriguing questions relates to the role and structure of dissipative, "viscous" effects, which are usually present as higher (usually second) order partial derivative terms in viscous regularizations of the hyperbolic conservation laws. Being typically small, those dissipative effects become important near shock waves or areas with sharp spatial gradients. The question asked by Gelfand, sounds deceptively simple: What structure of dissipative terms will ensure the convergence of solutions of the regularized hyperbolic conservation systems to those of the corresponding inviscid ones?

We consider strictly hyperbolic systems, whose Jacobians have distinct real eigenvalues. For those systems an (obvious) requirement for the viscosity matrix is to be positive definite, that is, "viscous". Of course, the diagonal viscosity matrices are the simplest, and correspond to the viscous terms in the Navier-Stokes equations of fluid motion. A diagonal viscous matrix with positive entries was used by Bianchini and Bressan [1] to prove the global existence and uniqueness for one-dimensional (1-D) strictly hyperbolic systems via the vanishing viscosity method.

In [10], Shizuta and Kawashima generalized the 1-D results to multidimensional linear systems with both the flux and viscosity matrices being symmetric matrices. Of course, the viscosity matrix was assumed to be nonnegative definite in [10]. The obtained result was applied to the discrete Boltzmann equation. Nevertheless, in the same year (1985), Majda and Pego [9] discovered a remarkably simple counter-intuitive phenomenon: viscous shocks for 1-D systems of convection-diffusion systems with strictly hyperbolic convective term and small and positive definite, but *non*-diagonal viscosity matrix may *not* converge to the shock waves of the corresponding inviscid system. Moreover, such system may be linearly unstable. Those instability examples have direct pertinence to the modeling of natural phenomena in larger, "macro" scales when both viscous and convective terms are not expected to be simply diagonal.

In [9], Majda and Pego considered the simplest 1-D linear system with constant coefficients,

$$\boldsymbol{u}_t + A\boldsymbol{u}_x = C\boldsymbol{u}_{xx}, \tag{1.1}$$

where x is a spatial variable, t is time, $\boldsymbol{u} \in \mathbb{R}^N$ is the vector of unknowns ($N \geq 2$), the matrix A has distinct real eigenvalues, and the matrix C has real positive eigenvalues. Even though one may expect the solutions of (1.1) to be smooth and stable, Majda and Pego observed that the simplest solution, $\boldsymbol{u}(x,t) = e^{ikx+\omega t}\hat{\boldsymbol{u}}$ with $\hat{\boldsymbol{u}} = \textbf{Const}$ and the linear stability operator $\Omega(k) := -ikA - k^2 C$ may have nontrivial behavior. It is obvious that for small k (the so-called long-wavelength solutions), the convective part is dominating and, in particular, the wave speeds are defined by the eigenvalues of A. This is similar to the case of a scalar convection-diffusion equation. However, unlike the scalar case, the stability of the system (1.1) is ensured by the negativity of the real parts of the eigenvalues of the matrix Ω, which is not guaranteed by the positivity of the eigenvalues of C, which dominates the eigenvalue problem for large k only. The behavior of the eigenvalues of $\Omega(k)$ for small k may be understood by the perturbative analysis of the simple "wavy" eigenvalues of the matrix A. Those perturbative terms include interplay between the matrices A and C. It turns out that Ω may have eigenvalues with positive real part for small wavenumbers k so that the corresponding solutions of (1.1) will grow indefinitely in time, contrary to the "naive" expectations. This may happen when C is positive definitive but not diagonal. We stress that the non-diagonality of both the advective matrix A and the viscosity matrix C is critical for the linear instability and that the results of Shizuta and Kawashima are not applicable in such case as only with symmetric matrices A and C were considered in [10].

Later, Kliakhandler [8] found the same unusual phenomenon for several very simple fluid mechanics problems, such as multi-layer laminar parallel flows. The outcome was unexpected and counter-intuitive: the effects that usually thought as dissipative, such as stable density stratification or surface tension, may actually have destabilizing effect on the shear flow. In the industry, the multi-layer laminar parallel flows have been extensively used in manufacturing of color photographic films. Laminar flows are usually very stable, so such a conclusion violated naïve expectations about flow behavior. As the results obtained in [8] were hard to believe, one of the authors of the present paper contacted Dr. Steven Weinstein, who worked at Kodak at that time. Dr. Steven Weinstein confirmed that such instabilities were often observed in the manufacturing of photographic films, and Kodak worked hard to avoid them to make the photo sensitive layers uniform.

Concerning the multidimensional systems, except for the case of symmetric fluxes and dissipative effects studied in [10], very little has been known about the influence of viscous terms on stability and convergence. It should be pointed out that in the vast majority of systems of PDEs appearing in various applied fields, the fluxes are *non*-symmetric. Moreover, dissipative

terms often do not constitute simple diagonal or symmetric matrices as well. Complex chemical and physical systems provide all sorts of fluxes and dissipation combinations, so simple Fick's law is not always applicable.

In this paper, we discover the two-dimensional (2-D) phenomenon that extends, in a certain sense, the 1-D results of Majda and Pego [9]. To this end, we consider the 2-D extension of (1.1),

$$\boldsymbol{u}_t + A\boldsymbol{u}_x + B\boldsymbol{u}_y = C\boldsymbol{u}_{xx} + D\boldsymbol{u}_{yy}, \tag{1.2}$$

where the advective matrices A and B have distinct real eigenvalues and the viscosity matrices C and D are positive definite, and show that viscous regularizations of 2-D strictly hyperbolic systems may be stable in the directions of both x- and y-axes, but yet some oblique waves may be unstable. We present an explicit example of such system (linearized shallow water equations), for which we perform linear stability analysis. We also conduct numerical simulations for this unstable linear system and also for nonlinear regularized Saint-Venant system. Our numerical results fully corroborate the obtained linear stability results. Finally, we conjecture on the stability of the eigenvalue problem and point to the obvious three-dimensional (3-D) extensions of our results.

2   Two-Dimensional Saint-Venant Systems

The Saint-Venant system introduced in [4] is one of the most commonly used models of shallow water flows in rivers or coastal areas.

In the case of flat bottom topography, the 2-D Saint-Venant system reads as

$$\begin{cases} h_t + q_x + p_y = 0, \\ q_t + \left(\dfrac{q^2}{h} + \dfrac{g}{2}h^2\right)_x + \left(\dfrac{pq}{h}\right)_y = 0, \\ p_t + \left(\dfrac{pq}{h}\right)_x + \left(\dfrac{p^2}{h} + \dfrac{g}{2}h^2\right)_y = 0. \end{cases} \tag{2.1}$$

Here, h is the water depth, u and v are the x- and y-velocities, respectively, $q = hu$ and $p = hv$ are the corresponding discharges, and g is the acceleration due to gravity.

The 2-D Saint-Venant system (2.1) can be linearized about a "lake-at-rest" steady state $h(x,y,t) \equiv h_0 = \texttt{Const}$, $u(x,y,t) = v(x,y,t) \equiv 0$. To this end, we introduce the perturbation variables h', q', and p' such that $h = h_0 + h'$, $q = q'$, and $p = p'$, and substitute them into (2.1) to obtain

$$\begin{cases} h'_t + q'_x + p'_y = 0, \\ q'_t + \left(\dfrac{(q')^2}{h_0 + h'} + \dfrac{g}{2}(h_0 + h')^2\right)_x + \left(\dfrac{p'q'}{h_0 + h'}\right)_y = 0, \\ p'_t + \left(\dfrac{p'q'}{h_0 + h'}\right)_x + \left(\dfrac{(p')^2}{h_0 + h'} + \dfrac{g}{2}(h_0 + h')^2\right)_y = 0. \end{cases} \tag{2.2}$$

We then neglect the nonlinear terms in (2.2) and end up with the 2-D linearized Saint-Venant system:

$$\begin{cases} h'_t + q'_x + p'_y = 0, \\ q'_t + gh_0 h'_x = 0, \\ p'_t + gh_0 h'_y = 0, \end{cases}$$

which can be rewritten in the vector form

$$\boldsymbol{u}_t + A\boldsymbol{u}_x + B\boldsymbol{u}_y = \boldsymbol{0}, \tag{2.3}$$

with

$$\boldsymbol{u} = \begin{pmatrix} h' \\ q' \\ p' \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ gh_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ gh_0 & 0 & 0 \end{pmatrix}. \tag{2.4}$$

3 Viscous Approximations of the Saint-Venant Systems

We now add the viscosity terms to the right-hand sides (RHS) of (2.3)–(2.4) and study the stability of the resulting viscous linearized shallow water equations, which read as (1.2), (2.4). We consider the following specific viscosity matrices C and D:

$$C = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}, \quad D = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ \Delta & \Delta & \varepsilon \end{pmatrix}, \tag{3.1}$$

where $\varepsilon > 0$ and Δ are constants.

We now analyze the stability of the linearized viscous shallow water equations (1.2), (2.4), (3.1). To this end, we take the following ansatz:

$$\begin{pmatrix} h' \\ q' \\ p' \end{pmatrix} = \begin{pmatrix} \hat{h}'_1 \\ \hat{q}'_1 \\ \hat{p}'_1 \end{pmatrix} e^{\omega t + ik\left(\gamma x + \sqrt{1-\gamma^2}\,y\right)}, \tag{3.2}$$

where ω is a wave magnitude, k is its frequency, γ stands for a spatial direction of the wave, and $(\hat{h}'_1, \hat{q}'_1, \hat{p}'_1)^\top$ is a constant nonzero vector. Substituting (3.2) into (1.2), (2.4), (3.1) results in

$$E\boldsymbol{u} = \boldsymbol{0},$$

where

$$E = \begin{pmatrix} \omega + \varepsilon k^2 \gamma^2 + \varepsilon k^2(1-\gamma^2) & ik\gamma & ik\sqrt{1-\gamma^2} \\ ik\gamma gh_0 & \omega + \varepsilon k^2 \gamma^2 + \varepsilon k^2(1-\gamma^2) & 0 \\ ik\sqrt{1-\gamma^2}gh_0 + \Delta k^2(1-\gamma^2) & \Delta k^2(1-\gamma^2) & \omega + \varepsilon k^2 \gamma^2 + \varepsilon k^2(1-\gamma^2) \end{pmatrix}.$$

Therefore, $\boldsymbol{u}$ is an eigenvector that corresponds to the zero eigenvalue if and only if the following characteristic equation is satisfied:

$$\varepsilon^3 k^6 - \Delta k^3 (1-\gamma^2)^{\frac{3}{2}}(gh_0 k\gamma + i\omega) + \varepsilon k^2(gh_0 k^2 - i\Delta k^3(1-\gamma^2)^{\frac{3}{2}} + 3\omega^2)+$$
$$+3\varepsilon^2 k^4 \omega + (gh_0 k^2 + \omega^2)\omega = 0.$$

Its solution ω depends on the wave number k and the parameters ε, g, h_0, and Δ. Clearly, if ω is positive for some k, then the system (1.2), (2.4), (3.1) is unstable.

Next, we take following parameter values: $h_0 = 1$, $g = 10$, $\Delta = 5$, and $\gamma = 1/2$, and then plot the function $\omega(k)$ for $\varepsilon = 1$ and $\varepsilon = 5$ in Figure 3.1. As one can see, when $\varepsilon = 5$, $\omega(k) \le 0$ for all k and thus the system is stable. However, when $\varepsilon = 1$, $\omega(k) > 0$ for $0 < k \lesssim 2.3$ with the maximum value about 0.4 achieved at about $k = 1.5$. This means that $\varepsilon = 1$ corresponds to the unstable regime.
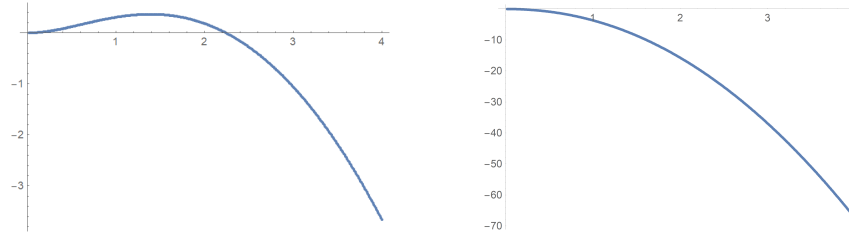

FIG. 3.1. w as a function of k for $\varepsilon = 1$ (left) and $\varepsilon = 5$ (right).

4   Numerical Examples

In this section, we conduct several numerical experiments using a semi-discrete fifth-order finite-difference alternative weighted essentially non-oscillatory (A-WENO) scheme described in [3, §3] (see also [2, Appendix C]). The diffusion terms have been discretized using a standard six-order finite-difference approximations of $\boldsymbol{u}_{xx}$ and $\boldsymbol{u}_{yy}$. We have used the three-stage third-order strong stability preserving (SSP) Runge-Kutta method (see, e.g., [6,7]) for the temporal discetization.

In all of the numerical examples, periodic boundary conditions are imposed on the four sides of the computational domain $[-30, 30] \times [-30, 30]$, which corresponds to 15 wavelengths in both the x- and y-directions. We compute the numerical solutions until a very large final time $t = 250$ on a uniform mesh with $\Delta x = \Delta y = 0.2$.

4.1. Linear Case.   We first conduct numerical experiments for the viscous linearized shallow water equations (1.2), (2.4), (3.1) subject to the following

initial conditions:

$$h'(x, y, 0) = \begin{cases} 10^{-2}\Big(1 + \dfrac{1}{40}e^{2(1-x^2-y^2)}\Big) & \text{if } x^2 + y^2 < 1, \\ 10^{-2}, & \text{otherwise,} \end{cases}$$

$$q'(x, y, 0) = p'(x, y, 0) \equiv 0,$$

which are a perturbation of the steady state $(h', q', p') \equiv (10^{-2}, 0, 0)$.

In Figures 4.1 and 4.2, we present the computed solutions for $\varepsilon = 5$ and 1, respectively. One can clearly see that the obtained numerical solutions support the analytical results in §3. Namely, when $\varepsilon = 5$, the solution is stable and the small initial perturbation decays in time as expected; see Figure 4.1, where we plot the computed solution at times $t = 0$, 50, 100, and 150. When $\varepsilon = 1$, the solution is unstable: It creates large magnitude wave structures in the oblique direction that corresponds to $\gamma = 1/2$, and the magnitude of these waves grows indefinitely in time; see Figure 4.2, where we plot the computed solution at times $t = 0$, 50, 100, 150, 200, and 250.
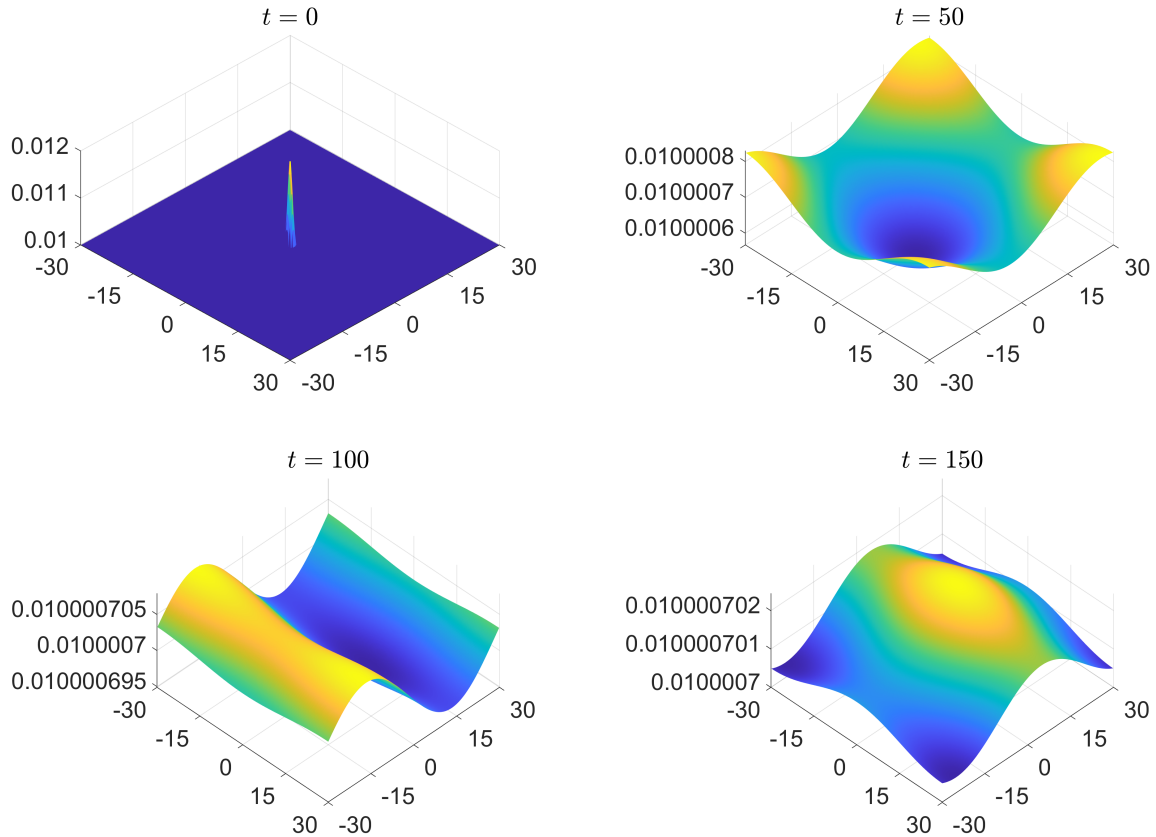

FIG. 4.1. Linear case: h computed with $\varepsilon = 5$ (stable regime) at different times.

$t = 0$

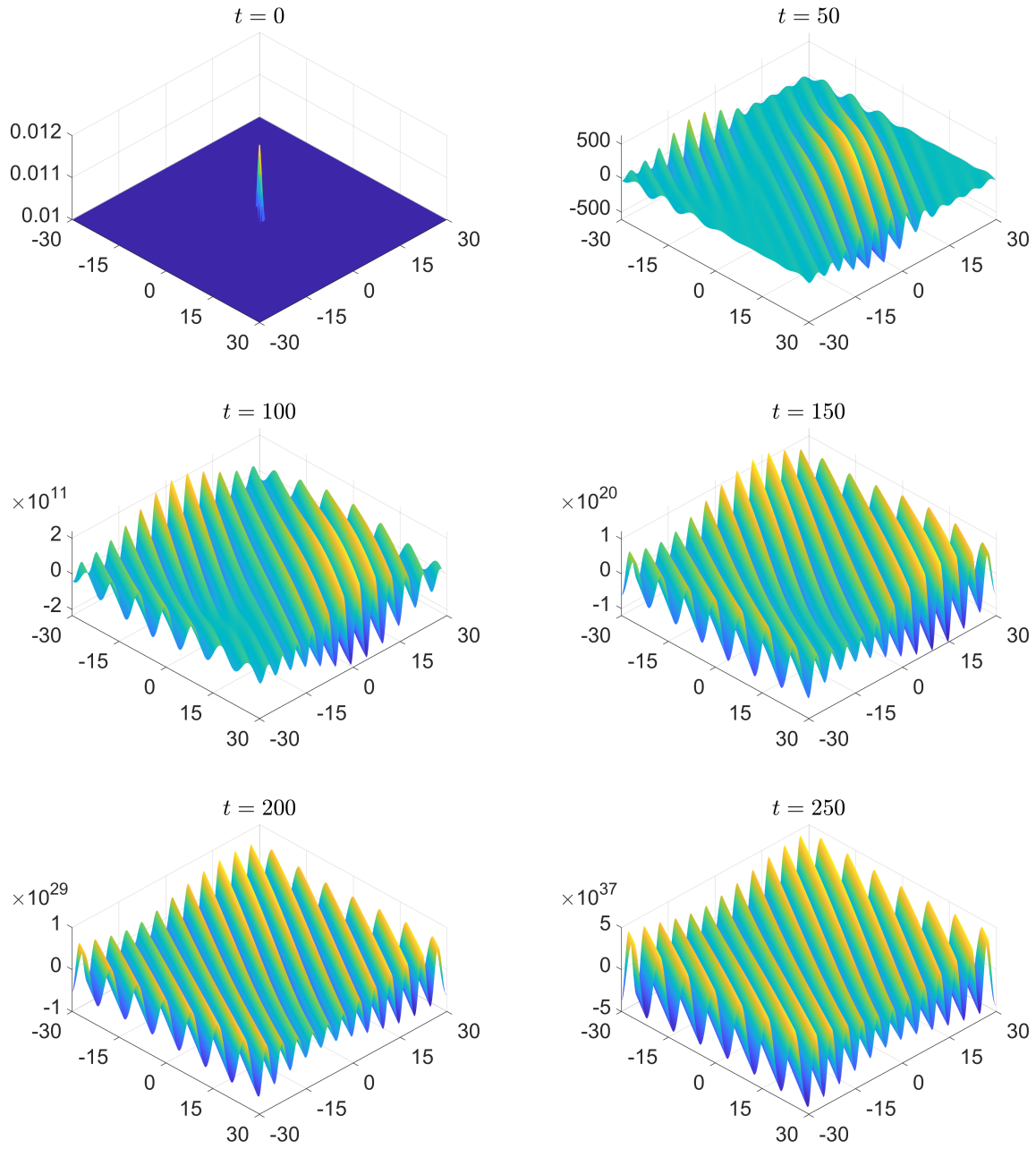$t = 50$

$t = 100$

$t = 150$

$t = 200$

$t = 250$

FIG. 4.2. Same as in Figure 4.1, but for $\varepsilon = 1$ (unstable regime).

4.2. Nonlinear Case.

In this section, we conduct numerical experiments for the nonlinear viscous Saint-Venant system

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{U})_x + \boldsymbol{g}(\boldsymbol{u})_y = C\boldsymbol{u}_{xx} + D\boldsymbol{u}_{yy}.$$

where

$$\boldsymbol{u} = \begin{pmatrix} h \\ q \\ p \end{pmatrix}, \quad \boldsymbol{f}(\boldsymbol{u}) = \begin{pmatrix} q \\ \dfrac{q^2}{h} + \dfrac{g}{2}h^2 \\ \dfrac{pq}{h} \end{pmatrix}, \quad \boldsymbol{g}(\boldsymbol{u}) = \begin{pmatrix} p \\ \dfrac{pq}{h} \\ \dfrac{p^2}{h} + \dfrac{g}{2}h^2 \end{pmatrix},$$

and the matrices C and D are given by (3.1). We take the following initial conditions:

$$h(x, y, 0) = \begin{cases} 1 + \dfrac{1}{4000}e^{2(1-x^2-y^2)} & \text{if } x^2 + y^2 < 1, \\ 1, & \text{otherwise,} \end{cases}, q(x, y, 0) = p(x, y, 0) \equiv 0,$$

which are a small perturbation of the steady state $(h, q, p) \equiv (1, 0, 0)$.

We present the computed solutions in Figures 4.3 and 4.4 for $\varepsilon = 5$ and 1, respectively. As one can see, when $\varepsilon = 5$ the solution of the nonlinear system is stable and, like in the linear case, the initial perturbation decays in time; see Figure 4.3, where we plot the computed solution at times $t = 0$, 50, 100, and 150. The solution behavior in the unstable ($\varepsilon = 1$) regime is, however, different from the linear case. While the solution develops large magnitude waves in the same oblique direction corresponding to $\gamma = 1/2$, the magnitude of these waves does not increase beyond 2 and the solution structure evolves in a rather complicated nonlinear manner; see Figure 4.4, where we plot the computed solution at times $t = 0$, 50, 100, 150, 200, and 250.

5   Stability Conjectures

In this section, we discuss a general question on conditions on the matrices A, B, C, and D for the stability of the system (1.2). Those ideas are prompted by simple "combinatorial" observations of 1-D systems in the x- and y-directions, as well as by the examples of unstable oblique waves above. We surmise that linear operators, stemming from all combinations of convective terms in all directions with viscous terms in all directions, should be stable for all wavenumbers.

Conjecture 5.1. *If the eigenvalues of the stability operators $\Omega_1(k) = -ikA - k^2C$, $\Omega_2(k) = -ikA - k^2D$, $\Omega_3(k) = -ikB - k^2C$, and $\Omega_4(k) = -ikB - k^2D$ are negative for all k, then the system (1.2) is stable.*

Remark 5.2. *In fact, Conjecture 5.1 suggests to check the eigenvalues of four possible combinations $-ik(\cdot) - k^2(\cdot)$ of all convective and viscous matrices.*

Before presenting the second conjecture, we will make a few simple observations. We first consider a general eigenvalue problem prompted by the stability considerations. The eigenvalues of A are the roots of the characteristic polynomial of degree N, $P_N(\lambda) = \sum_{i=0}^{N}(-1)^i r_i \lambda^{N-i}$ with $r_0 = 1$, $r_1 = \text{tr}(A)$, and $r_N = \det(A)$. In fact, the coefficients r_i for $i = 1, \dots, N$ are the sum of the
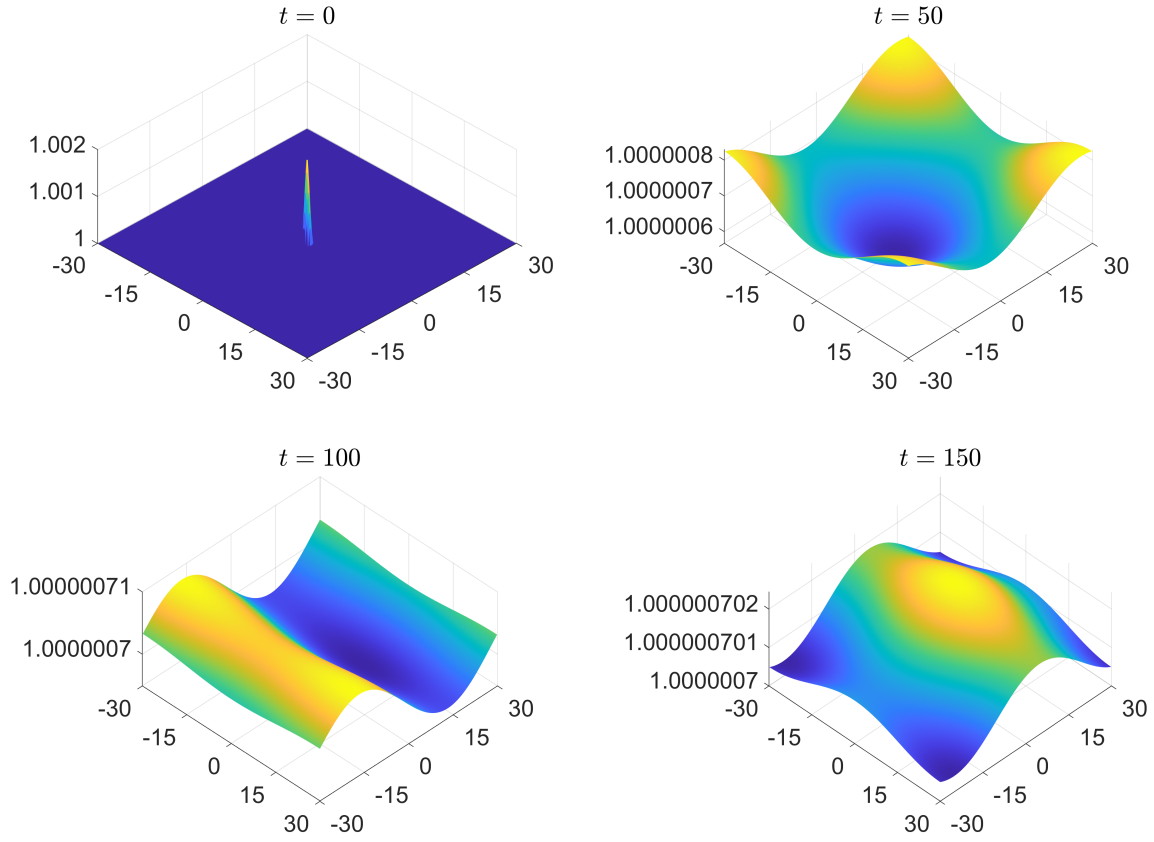
FIG. 4.3. Nonlinear case: h computed with $\varepsilon = 5$ (stable regime) at different times.

i-rowed principal minors of A. In particular, for $N = 2$ and $N = 3$, we have

$$P_2(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A) = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}),$$
$$P_2'(\lambda) = 2\lambda - \text{tr}(A),$$

$$P_3(\lambda) = \lambda^3 - \text{tr}(A)\lambda^2 + \Big(\sum_{i=1}^{3} M_{ii}\Big)\lambda - \det(A),$$

$$P_3'(\lambda) = 3\lambda^2 - 2\,\text{tr}(A)\lambda + \sum_{i=1}^{3} M_{ii}.$$

Here, M_{ij} denotes the (i, j) minor of A, and $\sum_{i=1}^{3} M_{ii} = (a_{11}a_{22} - a_{12}a_{21}) + (a_{11}a_{33} - a_{13}a_{31}) + (a_{22}a_{33} - a_{23}a_{32})$.

Let us denote by λ_ℓ, $\ell = 1, \ldots, N$ the eigenvalues of A and consider a slightly *perturbed* matrix $A + \delta C$, where δ is a small parameter. Its eigenvalues

$t = 0$

$t = 50$

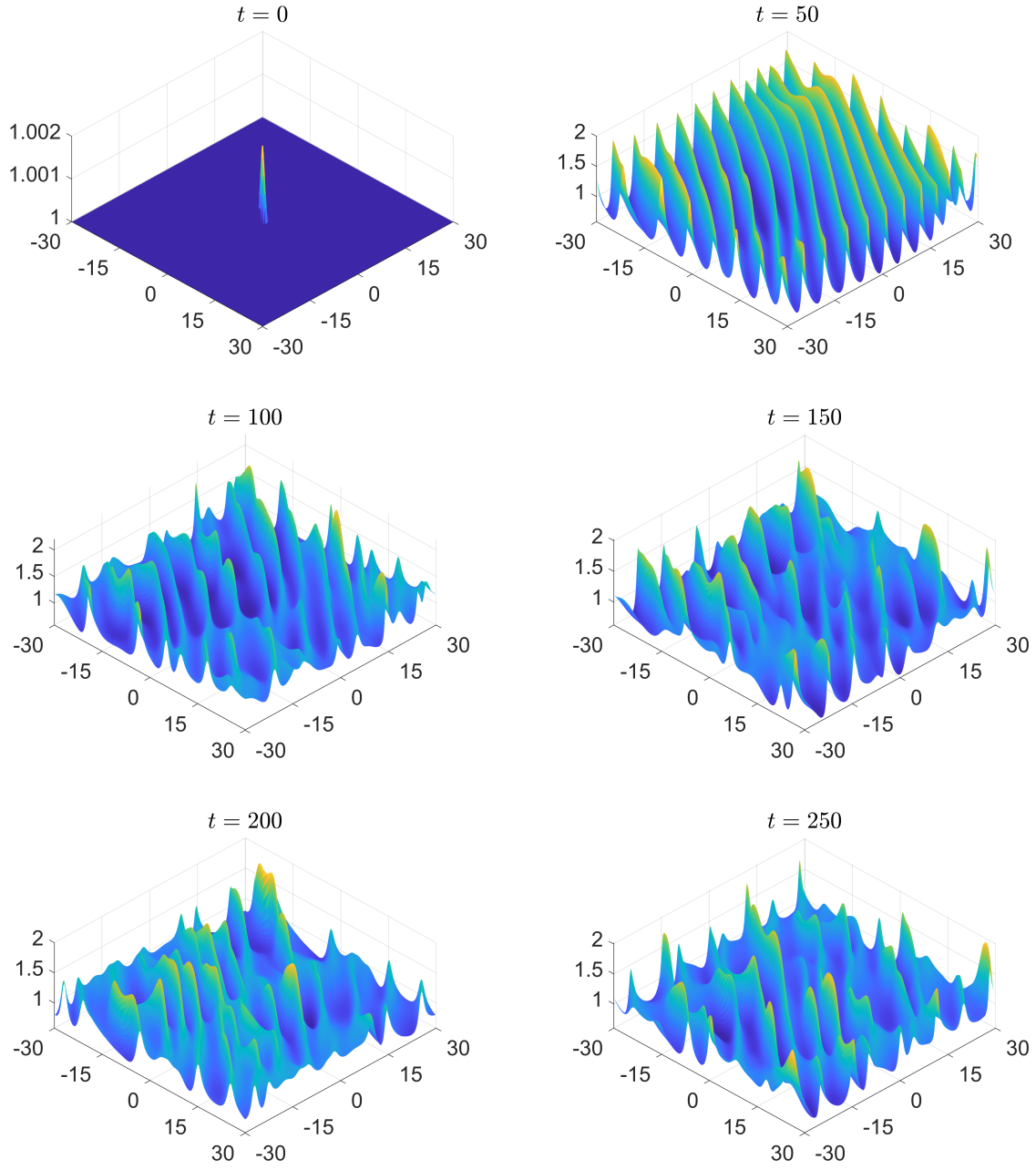$t = 100$

$t = 150$

$t = 200$

$t = 250$

FIG. 4.4. *Same as in Figure* 4.3, *but for* $\varepsilon = 1$ *(unstable regime).*

are $\lambda_\ell + \delta\lambda_{\ell,1} + \delta^2\lambda_{\ell,2} + \dots$ and the characteristic equation can be written as

$$\widetilde{P}_N(\lambda_\ell; \delta) := \sum_{i=0}^{N}(-1)^i(r_i + \delta r_{i,1} + \delta^2 r_{i,2} + \dots)(\lambda_\ell + \delta\lambda_{\ell,1} + \delta^2\lambda_{\ell,2} + \dots)^{N-i} = 0$$

$$(5.1)$$

with the coefficients on the left-hand side being slightly perturbed sums of all principal minors of $A + \delta C$. Differentiating (5.1) with respect to δ and substituting $\delta = 0$ results in

$$\frac{\partial \widetilde{P}_N}{\partial \delta}(\delta = 0) = \lambda_{\ell,1} P'_N(\lambda_\ell) + \sum_{i=1}^{N} (-1)^i r_{i,1} \lambda_\ell^{N-i} = 0,$$

which implies that the linear correction of the eigenvalue λ_ℓ, which may influence the sign of the ℓ-th eigenvalue of the perturbed matrix for small δ, is

$$\lambda_{\ell,1} = -\frac{1}{P'_N(\lambda_\ell)} \sum_{i=1}^{N} (-1)^i r_{i,1} \lambda_\ell^{N-i}, \quad \ell = 1, \ldots, N. \tag{5.2}$$

Here, $r_{i,1}$ are the sum of the i-rowed principal minors with all combinations of elements where precisely one element of matrix A is replaced by elements of matrix C (first-order correction) so that $\lambda_{\ell,1}$ is a linear function relative to the elements of matrix C. Notice that $P'_N(\lambda_\ell) \neq 0$ as long as the eigenvalues of A are distinct.

In particular, for $N = 2$, the first correction (5.2) assumes a simple elegant form:

$$\lambda_{\ell,1} = \frac{(c_{11} + c_{22})\lambda_\ell + a_{12}c_{21} - a_{22}c_{11} + a_{21}c_{12} - a_{11}c_{22}}{2\lambda_\ell - (a_{11} + a_{22})}, \quad \ell = 1, 2.$$

For $N = 3$, (5.2) takes the following (a bit cumbersome) form:

$$\lambda_{\ell,1} = \frac{\text{tr}(C)\lambda_\ell^2 - r_{2,1}\lambda_\ell + \sum_{i=1}^{3}\sum_{j=1}^{3}(-1)^{i+j}c_{ij}M_{ij}}{3\lambda_\ell^2 - 2\,\text{tr}(A)\lambda_\ell + \sum_{i=1}^{3}M_{ii}}, \quad \ell = 1, 2, 3,$$

where

$$r_{2,1} = a_{22}c_{33} + a_{33}c_{22} - a_{23}c_{32} - a_{32}c_{23} + a_{11}c_{33} + a_{33}c_{11} -$$
$$-a_{13}c_{31} - a_{31}c_{13} + a_{11}c_{22} + a_{22}c_{11} - a_{12}c_{21} - a_{21}c_{12}$$

We now use (5.2) and introduce the second conjecture, which relates to a more algebraic reduction of Conjecture 5.1.

Conjecture 5.3. *Let the matrix A has distinct real eigenvalues λ_ℓ, $\ell = 1, \ldots, N$, and the matrix C has real positive eigenvalues. Then, the linear operator $\Omega(k) = -ikA - k^2C$ will have negative eigenvalues if and only if the quantities $\lambda_{\ell,1}$ defined in (5.2) are negative for all $\ell = 1, \ldots, N$.*

We stress that values $\lambda_{\ell,1}$ control the first correction of long-wavelength expansion for small k: $-i\lambda_\ell k + \lambda_{\ell,1} k^2$. The importance of Conjecture 5.3 is that it gives a simple algebraic stability criterion based on the elements of matrices A and C for *all* k. Both Conjectures 5.1 and 5.3 may be viewed as

a direct and simple solution of the multidimensional Gelfand problem, and
they are inspired by the remarkable paper by Majda and Pego [9].

6   Conclusion

We have demonstrated the counter-intuitive extension of Majda-Pego insta-
bility for 2-D systems, both analytically and through direct numerical simu-
lations. To the best of our knowledge, this is the first nontrivial result for
the Gelfand problem in the multidimensional case.

While we have presented examples of unstable 2-D system with viscous
terms, it is rather easy to construct similar examples in the 3-D case: as in
the 2-D case, both advective and viscous matrices should be non-diagonal,
all of the waves propagating in either x-, y-, or z-direction will be stable, yet
some oblique waves will be unstable.

We hope that our examples will stimulate the elucidation of peculiar role of
viscous effects in the broad spectrum of natural phenomena, including fluid
mechanics. We have proposed the conjectures that give direct assessment
of stability of multidimensional convection-diffusion systems and resolve the
Gelfand problem. One can also generalize Conjectures 5.1 and 5.3 for the 3-D
systems: In order to establish the stability criteria, one has to study to the
eigenvalues of nine linear operators $-ik(\cdot) - k^2(\cdot)$ for possible combination
of all convective and viscous matrices.

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