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APPLICATION OF LINEAR FRACTIONAL TRANSFORMATION IN PROBLEMS OF LOCALIZATION OF MATRIX SPECTRA AND ROOTS OF POLYNOMIALS

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Dedicated to the memory of Sergey Godunov

Abstract: The paper investigates the possibilities of using linear fractional transformations in a number of problems that can be reduced to spectral dichotomy. More specifically, for the dichotomy of the imaginary axis, estimates are given for areas containing eigenvalues, methods for determining the absence of a matrix spectrum on a ray and a segment are described. A method for dividing a polynomial into two factors whose roots lie in the right and left half-planes is described and substantiated.

Keywords: spectrum dichotomy, linear fractional transformation, factorization of a polynomial.

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Introduction

This work is devoted to the development of S.K. Godunov's ideas in the field of studying matrix spectra. The peculiarity of his approach to creating new numerical methods was the assumption that the ill-conditioning of a mathematical problem may be due to the fact that the properties of the original physical problem are not sufficiently taken into account when it is stated. That is, it is the statement of the problem that first of all requires revision. Such a view of the problem allowed, for example, to create a method for regularizing ill-conditioned SLAEs using additional equations indicating the smoothness of solutions [1] (Chapter 7, Section 8).

A similar approach can be traced in the matrix spectrum dichotomy method. It is known that the numerical solution of an asymmetric spectral problem in the classical formulation can be extremely sensitive to the accuracy of the matrix specification and computational errors (see, for example, [2], paragraph 1.3). At the same time, the formulation of the dichotomy problem more accurately reflects the properties of a number of practical problems, among which, in particular, is the stability problem. Indeed, the essence of the spectrum dichotomy is to calculate for a given matrix pencil $A - \lambda B$ and a fixed curve γ a self-adjoint positive definite matrix $H_\gamma(A, B)$, whose norm $\omega_\gamma(A, B) = \|H_\gamma(A, B)\|$ is a numerical criterion for the dichotomy. This means that if the value $\omega_\gamma(A, B)$ is not too large $\omega_\gamma(A, B) < \omega_{\max}$, then some neighborhood of the curve γ is free of points of the spectrum of the pencil $A - \lambda B$. Moreover, in a number of cases of specific curves, an explicit estimate can be given for the width of this neighborhood. If the absence of eigenvalues on γ is thus established, then the entire space is divided into two eigensubspaces corresponding to points of the spectrum lying on either side of the curve γ. In the process of executing the dichotomy algorithms, in parallel with the calculation of the matrix criterion $H_\gamma(A, B)$, the projectors $P_\gamma(A, B)$ and $I - P_\gamma(A, B)$ onto these subspaces (here I is the identity matrix) are calculated. Recall that the dimension of each subspace, and therefore the number of eigenvalues that are on one side of γ, coincides with the trace of the projector.

Thus, the dichotomy method allows one to determine how many eigenvalues of the matrix lie in the regions separated by a given γ curve, for example, in the stability and instability regions, the bases of the corresponding invariant spaces, and in some cases to estimate the distance from the spectrum to the curve. These are exactly the data that are required to solve Lyapunov stability problems. In a certain sense, the spectrum dichotomy is a natural continuation of Lyapunov stability theory for linear systems of differential and difference equations. In cases where γ is a unit circle or imaginary axis, and the spectrum lies inside the circle or in the left half-plane, the matrix dichotomy criterion is the solution of the corresponding Lyapunov equation, which in turn is a matrix of a square form representing the Lyapunov function.

Within the group of dichotomy algorithms, the division of the spectrum by a unit circle plays a fundamental role (one of the variants of this algorithm is briefly described in the appendix). For example, the problem of dichotomy by an imaginary axis was usually reduced to the case of a circle using an exponential transformation [3], the use of which, however, has a number of limitations. In particular, in spectral problems obtained by discretizing differential operators, such an approach without special normalization and additional iterations always yields a false negative result. In [4] we showed that in order to reduce the problem to a dichotomy by a circle, a linear fractional transformation can be used, which does not have these limitations. Note that in the book [2] in section 10.3, the linear fractional transformation connecting problems in the half-plane and in the circle is mentioned, but is not used in the algorithms. In this paper, we expand the scope of applicability of linear fractional transformations to spectral problems. In the first section, the properties of the linear fractional transformation in the problem of dichotomy by the imaginary axis are studied in more detail. In the second section, the linear fractional transformation is used in the problem of checking the absence of a spectrum on a segment. And finally, in the third section, the problem of decomposing a polynomial into two factors, the roots of which lie in the right and left half-planes, respectively, is considered. This problem is also proposed to be solved using linear fractional transformations.

1 A new method of dichotomy with respect to the imaginary axis

1.1. Preliminary remarks. Note that the linear fractional transformation of the complex plane of the spectral parameter λ

$$\xi = \frac{1+\lambda}{1-\lambda}, \qquad \lambda = \frac{\xi - 1}{\xi + 1} \tag{1}$$

transforms the left half-plane into the unit circle. Accordingly, the problem of dividing the spectrum of the matrix pencil $A - \lambda B$ with respect to the imaginary axis is reduced to the problem of dividing with respect to the unit circle of spectrum of the pencil (for more details, see [4])

$$A_0 - \xi B_0, \qquad A_0 = A + B, \qquad B_0 = B - A. \tag{2}$$

In this case, the matrix criterion of dichotomy of the original problem H_{Im}

$$H_{Im}(A,B) = \int_{Re(\lambda)=0} (A - \lambda B_0)^{-1}(AA^* + BB^*)(A - \lambda B_0)^{-*} d|\lambda|. \tag{3}$$

differs from the criterion $H_{r=1}$ of the problem for the circle

$$H_{r=1}(A_0, B_0) = \frac{1}{2\pi}\int_0^{2\pi} (A_0 - e^{i\varphi}B_0)^{-1}(A_0A_0^* + B_0B_0^*)(A_0 - e^{i\varphi}B_0)^{-*} d\varphi$$

by a factor of 2: $H_{Im}(A,B) = 2H_{r=1}(A_0, B_0)$.

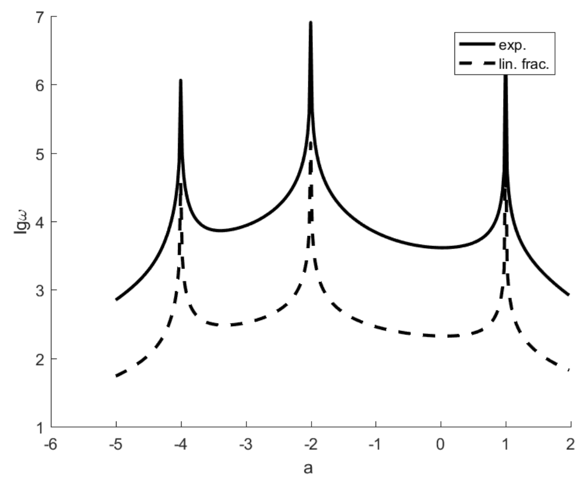

FIG. 1. Spectral portraits, comparison of two approaches

An obvious advantage of this approach is that there is no need to calculate the inverse matrix of B and then the matrix exponential $\exp(B^{-1}A)$.

Note that to separate the spectrum relative to a straight line, parallel to the imaginary axis $Re\lambda = a$, it is sufficient to perform a preliminary shift of the spectral parameter $A - (\lambda - a)B$ and, in fact, solve the problem of dichotomy by the imaginary axis for the pencil $(A + aI) - \lambda B$. The graphs of the functions $\omega(a) = \|H_{Im}(A - aI, B)\|$ are called one-dimensional spectral portraits, which allow us to visualize the location of the spectrum on the plane. For illustration, we present one-dimensional spectral portraits of the following matrix

$$A = \begin{bmatrix} 1 & -1 & 3 & 4 & 8 \\ 1 & 1 & 7 & 9 & 2 \\ 0 & 0 & -4 & -1 & 7 \\ 0 & 0 & 1 & -4 & 6 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}, \tag{4}$$

obtained using the traditional exponential and the linear fractional transformations described above (see Fig. 1).

The eigenvalues of the matrix lie on the lines $Re\lambda = a$, where $a = -4$, -2, 1. These values correspond to the "peaks" on the graphs of $\log_{10} \omega(a)$.

It is important that the norm of the dichotomy criterion obtained in the new way is smaller by orders of magnitude. This means that the linear fractional transformation can be successfully used in narrower neighborhoods of the eigenvalues, and also requires fewer iterations at the stage of dichotomy by a circle.

1.2. Regions containing eigenvalues. Suppose that the pencils $A - \lambda B$ and $A_0 - \xi B_0$ (2) have no eigenvalues on the imaginary axis and the unit

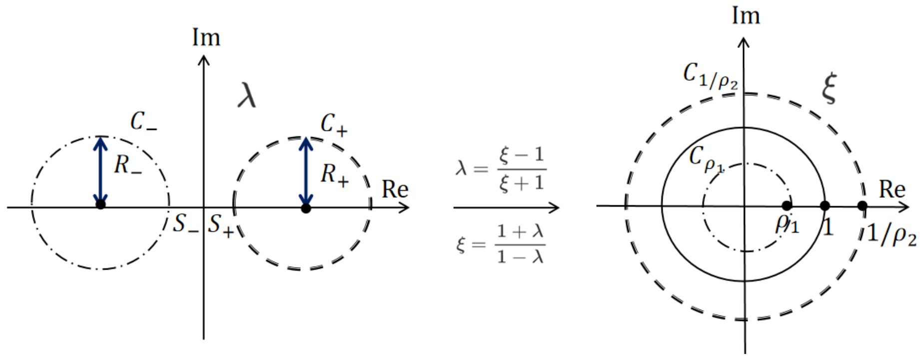

FIG. 2. Fractional linear transformation of the complex plane

circle, respectively. Then the eigenvalues of the pencil $A_0 - \xi B_0$ are also absent in the annulus between the circles C_{ρ_1} and C_{1/ρ_2} with center at zero and radii $0 < \rho_1 < 1$ and $1 < 1/\rho_2$ (see Fig. 2). Further, we assume that ρ_1 and ρ_2 denote the minimum possible values, that is, ρ_1 is the maximum modulus of the eigenvalues of the pencil $A_0 - \xi B_0$ that are inside the unit circle, $1/\rho_2$ is the minimum modulus of the eigenvalues that are outside the unit circle.

Consider the matrix Green's function, which is a solution to the boundary value problem

$$A_0 G_j - B_0 G_{j+1} = 0, \quad G_{+0} - G_{-0} = I, \quad \|G_j\| \xrightarrow[|j|\to\infty]{} 0.$$

Since the pencil $A_0 - \xi B_0$ is regular on the unit circle, the Green's matrix exists, is unique, and (as follows from the canonical decomposition of the pencil [2], Section 10.3) can be represented as

$$G_j = S^{-1} \begin{pmatrix} \Lambda^{|j|} & 0 \\ 0 & 0 \end{pmatrix} S \text{ for } j \geq +0, \quad G_j = S^{-1} \begin{pmatrix} 0 & 0 \\ 0 & M^j \end{pmatrix} S \text{ for } j \leq -0,$$

where $\det S \neq 0$, the set of eigenvalues of the pencil $A_0 - \xi B_0$ lying inside the unit circle coincides with the spectrum of Λ, and the inverses to the eigenvaluesof the pencil $A_0 - \xi B_0$ lying outside the unit circle coincide with the spectrum of M. Thus, the minimum possible value of ρ_1 is equal to the spectral radius of the matrix Λ or G_1, and ρ_2 is equal to the spectral radius of the matrices M, G_{-1}:

$$\rho_1 = \rho(\Lambda) = \rho(G_1) = \lim_{j\to\infty} \|G_1^j\|^{1/j} = \lim_{j\to\infty} \|G_j\|^{1/j},$$

$$\rho_2 = \rho(M) = \rho(G_{-1}) = \lim_{j\to\infty} \|G_{-1}^j\|^{1/j} = \lim_{j\to\infty} \|G_{-j}\|^{1/j}.$$

The following estimate holds:

$$\|G_{\pm j}\| \leq \sqrt{\|H\|}\varkappa^j, \text{ where } \varkappa = \sqrt{\frac{\|H\| - 1}{\|H\| + 1}}$$

(for the proof, see [2], Section 14.4). Here and below, for brevity, we introduce the notation $H = H_{r=1}(A_0, B_0)$. This implies the inequality for the spectral radius:

$$\rho(G_{\pm 1}) \le \lim_{j\to\infty} \left(\sqrt{\|H\|}\varkappa^j\right)^{\frac{1}{j}} = \lim_{j\to\infty} (\sqrt{\|H\|})^{1/j}\varkappa = \varkappa,$$

and hence the estimate

$$\rho_{1,2} \le \varkappa. \tag{5}$$

Now let us consider circles C_- of radius R_- and C_+ of radius R_+, lying in the left and right half-planes on the plane of the parameter λ, which are the preimages of circles C_{ρ_1} and C_{1/ρ_2} under the transformation (1). From the injectivity of the transformation it follows that the entire spectrum of the original pencil $A - \lambda B$, provided that the criterion of the dichotomy by the imaginary axis is bounded, is divided into two parts, one of which lies strictly inside the circle C_-, and the other inside C_+.

Let us also denote by S_- and S_+ the distances from the imaginary axis to circles C_-, C_+, respectively. Obviously, the distance from the spectrum of the pencil $A - \lambda B$ to the imaginary axis exceeds $\min\{S_1, S_2\}$.

Theorem 1. *For the radii R_-, R_+ of the circles C_-, C_+ and the distances S_-, S_+ from them to the imaginary axis, the following estimates hold*

$$R_\pm \le \sqrt{\|H\|^2 - 1}, \qquad S_\pm \ge \|H\| - \sqrt{\|H\|^2 - 1}, \tag{6}$$

where $H = H_{r=1}(A_0, B_0)$ is the criterion for the dichotomy of the spectrum of the pencil (2) by the unit circle .

Proof. Consider the circle C_{ρ_1}. Its equation is $\xi\bar{\xi} = \rho_1^2$. It is easy to establish that the equation of its preimage C_- has the form

$$\lambda\bar{\lambda} - \left(1 - \frac{2}{1-\rho_1^2}\right)\lambda - \left(1 - \frac{2}{1-\rho_1^2}\right)\bar{\lambda} + 1 = 0.$$

It follows that the center of this circle is at the real point $O_- = -(1 + \rho_1^2)/(1 - \rho_1^2)$, and the radius is $R_- = 2\rho_1/(1 - \rho_1^2)$. This means that the distance from the circle C_- to the imaginary axis is $S_- = (1 - \rho_1)/(1 + \rho_1)$. Similarly, we obtain $R_+ = 2\rho_2/(1 - \rho_2^2)$ and $S_+ = (1 - \rho_2)/(1 + \rho_2)$. Next, we use the estimate (5), as a result of which we arrive at inequalities (6). □

Corollary 1. *The distance between the eigenvalues of the matrix pencil $A - \lambda B$ and the imaginary axis exceeds the value $\|H\| - \sqrt{\|H\|^2 - 1}$.*

2 Checking the absence of eigenvalues of a matrix pencil on a ray and on a segment

2.1. The absence of eigenvalues on a ray. Let l be the ray with the origin at 0, and let the angle between it and the real positive semi-axis in the counterclockwise direction be α (see Fig. 3, left). Then, using rotation (multiplication by the complex number $e^{i\alpha}$), checking for the absence of the spectrum of the pencil $A - \lambda B$ on a given ray l is reduced to checking for the absence of the spectrum of the pencil $A_\alpha - \lambda B = e^{i\alpha} A - \lambda B$ on the real positive semi-axis R^+.

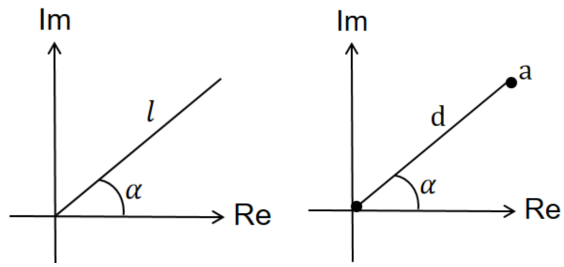

FIG. 3. Schemes for problems on the absence of a matrix spectrum on a ray (left), on the absence of a spectrum on a segment (right)

At the same time, among the eigenvalues of the matrix pencil $A_\alpha - \lambda B$ there are no positive real numbers if and only if among the eigenvalues of the quadratic matrix pencil $A_\alpha + \xi^2 B$ there are no imaginary numbers. In this case, the spectrum of this quadratic matrix pencil coincides with the spectrum of a linear pencil of twice the size

$$\hat{A} - \xi \hat{B} = \begin{bmatrix} A_\alpha & \\ & I \end{bmatrix} - \xi \begin{bmatrix} & -B \\ I & \end{bmatrix}, \tag{7}$$

where I is the identity matrix of the same size as the matrices A and B. This statement follows from the equality of the determinants

$$\det(A_\alpha + \xi^2 B) = \det\left(\begin{bmatrix} A_\alpha & \\ & I \end{bmatrix} - \xi \begin{bmatrix} & -B \\ I & \end{bmatrix}\right) = \det\left(\begin{bmatrix} A_\alpha & \xi B \\ -\xi I & I \end{bmatrix}\right).$$

If $B = I$, then, taking into account the following obvious equality

$$\det\left(\begin{bmatrix} A_\alpha & \\ & I \end{bmatrix} - \xi \begin{bmatrix} & -I \\ I & \end{bmatrix}\right) =$$

$$= \det\left(\begin{bmatrix} & -I \\ I & \end{bmatrix}\right) \det\left(\begin{bmatrix} & I \\ -A_\alpha & \end{bmatrix} - \xi \begin{bmatrix} I & \\ & I \end{bmatrix}\right),$$

we can conclude that the spectrum pencil (7) coincides with the spectrum of the matrix

$$\mathbf{A} = \begin{bmatrix} & I \\ -A_\alpha & \end{bmatrix} \tag{8}$$

Then, using a linear fractional transformation (see above), the problem is reduced to the question of the absence of eigenvalues on the unit circle.

Thus, if for the pencil (7) the criterion of the dichotomy of the spectrum by the imaginary axis is not too large, then it can be stated that on the ray l there are no eigenvalues of the pencil $A - \lambda B$.

Note that in this case, no projectors are calculated, since the ray does not divide the plane into two non-intersecting regions. But we can calculate the criterion of dichotomy by the imaginary axis $\omega_{Im}(\hat{A}, \hat{B}) = \|H_{Im}(\hat{A}, \hat{B})\|$ and use it as a criterion for the absence of a spectrum of the pencil $A - \lambda B$ on ray l: $\omega_l(A, B) = \omega_{Im}(\hat{A}, \hat{B})$.

2.2. The absence of eigenvalues on a segment. Next, we consider the segment $d = [0, a]$. In order to establish the absence of eigenvalues of the pencil $A - \lambda B$ on the given segment, we will use the linear fractional transformation $d \to R^+$,

$$\xi = \frac{\lambda}{\lambda - a}, \qquad \lambda = \frac{a\xi}{\xi - 1}. \tag{9}$$

In this case, the origin of coordinates (one end of the segment) goes into itself, and the other end of the segment is transformed into infinity.

Substitute the expression for λ (9) into the equality $\det(A - \lambda B) = 0$ and obtain a spectral problem with respect to the parameter ξ: $\det(A - \xi(A - aB)) = 0$. In this case, the location or absence of the eigenvalues of the matrix A on the segment $[0, a]$ is equivalent to the location or absence of the eigenvalues of the matrix pencil $A - \xi(A - aB)$ on the real positive semiaxis.

Further, as above, the problem is reduced to the dichotomy by the unit circle. As a result, the value $\omega_d(A, B) = \omega_{R^+}(A, A - aB)$ can be considered as the criterion for the absence of the spectrum of the pencil $A - \lambda B$ on the segment d.

Note. This approach can be used as a part of algorithms for separating spectra relative to a polygon to establish the absence of a spectrum on its sides.

2.3. Example. Consider a set of rays l_α and a set of segments d_α of length r, forming an angle α with the positive part of the real axis, $0 < \alpha \leq 2\pi$, each of which has one end coincident with the origin. The approach presented above can be applied to each of these rays and segments. As a result, the value $\omega_l(\alpha)$ and $\omega_d(\alpha)$ of the numerical criteria for the absence of eigenvalues of a given matrix A on a ray or on a segment will be obtained for each angle α.

Fig.4 shows the graphs of these functions on a logarithmic scale for the matrix (4) at $r = 3$.

On the graph of the function $\log_{10} \omega_l(\alpha)$, 5 peaks are visible, the arguments of which correspond to the angles of the rays passing through the eigenvalues in the following order $1 + i$, $-4 + i$, -2, $-4 - i$, $1 - i$, while on the graph $\log_{10} \omega_d(\alpha)$ there are only 3 peaks, since the moduli of the other eigenvalues exceed the given segment length $r = 3$.

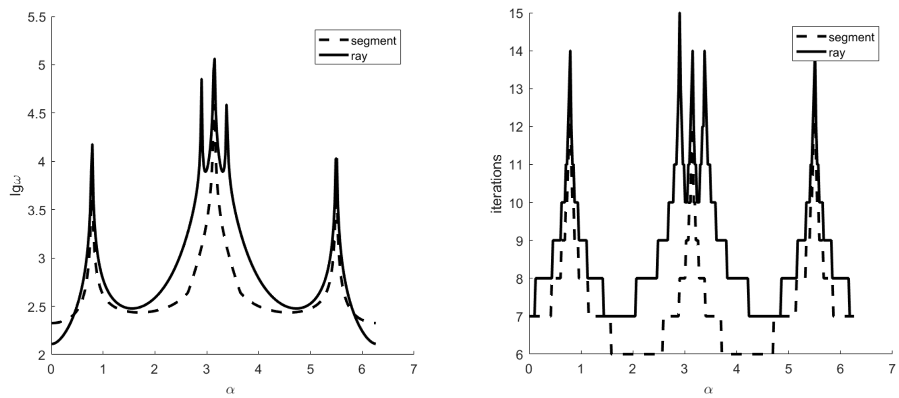

FIG. 4. The value of the criterion of absence of the matrix spectrum (4) on the ray $\omega_l(\alpha)$ and on the segment $\omega_d(\alpha)$ (left), the number of iterations required (right)

3 Decomposition of polynomials

3.1. Notation and formulation of the theorem. In this section, we consider the problem of factoring the polynomial $f(\lambda)$ into factors $g(\lambda)$, $h(\lambda)$

$$f(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n,$$
$$g(\lambda) = b_0 + b_1\lambda + b_2\lambda^2 + \cdots + b_l\lambda^l,$$
$$h(\lambda) = c_0 + c_1\lambda + c_2\lambda^2 + \cdots + c_m\lambda^m$$

such that l roots $g(\lambda)$ lies in the left half-plane, and m roots of $h(\lambda)$ lie in the right, $l + m = n$.

Let us introduce the following notation. Let the matrix A have the form

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \ddots & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \cdots & \cdots & -\frac{a_{n-1}}{a_n} \end{bmatrix},$$

where the elements of the bottom row are expressed in terms of the coefficients of the polynomial $f(\lambda)$. Let us construct a matrix pencil $A_0 - \xi B_0$ using the linear fractional transformation (1): $A_0 = A + I_n$, $B_0 = I_n - A$, where I_n is the identity matrix of size n.

Next, we recursively define matrices A_k, B_k ($k = 0, 1, \ldots$) using qr-decomposition

$$\begin{bmatrix} -B_k & A_k & 0 \\ A_k & 0 & -B_k \end{bmatrix} = Q_{k+1} \begin{bmatrix} * & * & * \\ 0 & A_{k+1} & -B_{k+1} \end{bmatrix}, \tag{10}$$

where $Q_{k+1}Q_{k+1}^* = I_{2n}$. Note that this recurrent transformation is the basis of the iterations of the unit circle dichotomy algorithm (see section 4).

Let us also denote by $\widetilde{A}_k$ and $\widetilde{B}_k$ the lower triangular matrices obtained as a result of the ql-decomposition of the matrices A_k, B_k. We use the superscript to denote the matrix row number so that $\widetilde{A}_k^{[l+1]}$ and $\widetilde{B}_k^{[m+1]}$ are the $l+1$-th and $m+1$-th rows of the matrices $\widetilde{A}_k$, $\widetilde{B}_k$. In addition, we denote the upper and lower blocks of the matrices $\widetilde{A}_k$, $\widetilde{B}_k$ as follows:

$$\widetilde{A}_k = \begin{bmatrix} \overline{A}_k \\ \underline{A}_k \end{bmatrix}, \qquad \widetilde{B}_k = \begin{bmatrix} \overline{B}_k \\ \underline{B}_k \end{bmatrix},$$

where the blocks $\overline{A}_k$ and $\underline{B}_k$ contain l rows, and $\underline{A}_k$ and $\overline{B}_k$ contain m rows.

In this notation, we formulate the theorem on which the proposed numerical method for dividing a polynomial into factors is based.

Theorem 2. *If the criterion for dichotomy by the unit circle of the pencil $A_0 - \xi B_0$ is finite, then*

$$\lim_{k\to\infty} \|\overline{A}_k\| = 0, \qquad \lim_{k\to\infty} \|\overline{B}_k\| = 0,$$
$$\lim_{k\to\infty} \widetilde{A}_k^{[l+1]} = (b_0, \dots, b_l, 0, \dots, 0),$$
$$\lim_{k\to\infty} B_k^{[m+1]} = (c_0, \dots, c_m, 0, \dots, 0),$$

where b_j, c_j are the sought coefficients of the factors $g(\lambda)$, $h(\lambda)$.

3.2. The proof of the theorem.

The proof of this theorem is based on the following statements.

Theorem 3. *Let the pencil $A_0 - \xi B_0$ have no eigenvalues on the unit circle, and for $k \geq 0$ the matrices A_k, B_k are defined by the relation (10). Then the pencil $A_k - \xi B_k$ has no eigenvalues on the unit circle. In this case, there exist nonsingular matrices T_k and S, as well as matrices Λ of size $l \times l$ and M of size $m \times m$, all of whose eigenvalues lie inside the unit circle, such that the following decomposition holds:*

$$A_k = T_k \begin{pmatrix} \Lambda^{2^k} & 0 \\ 0 & I_m \end{pmatrix} S, \qquad B_k = T_k \begin{pmatrix} I_l & 0 \\ 0 & M^{2^k} \end{pmatrix} S.$$

Here I_l and I_m are the identity matrices of size $l \times l$ and $m \times m$, respectively. The singular values and condition numbers of the matrices T_k are bounded:

$$\frac{\sigma_1(T)}{\Pi_{j=0}^{k}(1 + \max\{\|\Lambda^{2^k}\|, \|M^{2^k}\|\})} \leq \sigma_j(T_k) \leq \sigma_n(T).$$

The proof of theorem 3 is given in the appendix.

Corollary 2. *If the singular values $\sigma_j(A_k)$, $\sigma_j(B_k)$ are numbered in ascending order, then*

$$\sigma_j(A_k) \leq \|T_k\| \, \|S\| \, \|\Lambda^{2^k}\| \ \textit{at}\ j \leq l,$$
$$\sigma_1(T_k)\sigma_1(S) - \|T_k\| \, \|S\| \, \|\Lambda^{2^k}\| \leq \sigma_j(A_k) \leq \|T_k\| \, \|S\|(1 + \|\Lambda^{2^k}\|) \ \textit{for}\ j > l,$$

$$\sigma_j(B_k) \le \|T_k\|\,\|S\|\,\|M^{2^k}\| \ \ at\ j \le m,$$

$$\sigma_1(T_k)\sigma_1(S) - \|T_k\|\,\|S\|\,\|M^{2^k}\| \le \sigma_j(B_k) \le \|T_k\|\,\|S\|(1+\|M^{2^k}\|) \ \ for\ j > m.$$

Proof. The proof is based on the decompositions

$$A_k = T_k \begin{pmatrix} 0 & 0 \\ 0 & I_m \end{pmatrix} S + T_k \begin{pmatrix} \Lambda^{2^k} & 0 \\ 0 & 0 \end{pmatrix} S,$$

$$B_k = T_k \begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix} S + T_k \begin{pmatrix} 0 & 0 \\ 0 & M^{2^k} \end{pmatrix} S$$

and the estimate for the singular values of the matrix sum ([2], Section 7.5)

$$\sigma_j(A) - \|B\| \le \sigma_j(A+B) \le \sigma_j(A) + \|B\|.$$

□

The following statments are also quite obvious and are of an auxiliary nature.

Lemma 1. *Let $x_1, \dots, x_l$ be complex numbers, where $x_i \ne x_j$,*

$$X = \begin{pmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_l \\ \vdots & & \vdots \\ x_1^m & \dots & x_l^m \end{pmatrix}.$$

Let the limit $\lim_{k\to\infty} \|p^{[k]}X\| = 0$ also hold, where $p^{[k]}$ is a sequence of vectors $p^{[k]} = (p_0^{[k]}, \dots, p_m^{[k]})$. Then for $m \le l$ the sequence has a limit $p = 0$, and for $m = l+1$ under condition $\|p^{[k]}\| \ge c > 0$ there is a limit $p \ne 0$, where p_j are the coefficients of the polynomial

$$p(x) = p_0 + p_1 x + \dots + p_l x^l,$$

whose roots are all the numbers $x_1, \dots, x_l$.

Lemma 2. *Let $x_1, \dots, x_l$ be complex numbers, among which there are multiples. Let matrix X consist of columns of the form $(1, x_i, \dots, x_i^m)^T$ in case $x_i \ne x_j$, and of groups of columns*

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ x_i & 1 & \ddots & 0 \\ x_i^2 & 2x_i & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_i^{q-1} & (q-1)x_i^{q-2} & \ddots & (q-1)! \\ \vdots & \vdots & & \vdots \\ x_i^m & m x_i^{m-1} & \dots & \frac{m!}{(m-q)!} x_i^{m-q+1} \end{pmatrix}$$

if x_i has multiplicity q. Let the limit $\lim_{k\to\infty} \|p^{[k]}X\| = 0$ also hold, where $p^{[k]}$ is a sequence of vectors $p^{[k]} = (p_0^{[k]}, \dots, p_m^{[k]})$. Then for $m \le l$ the sequence has a limit $p = 0$, and for $m = l+1$ under condition $\|p^{[k]}\| \ge c > 0$ there is a limit $p \ne 0$, where p_j are the coefficients of the polynomial

$$p(x) = p_0 + p_1 x + \dots + p_l x^l,$$

whose roots are all the numbers $x_1, \dots, x_l$, taking into account their multiplicity.

Lemma 3. *Let $x_1(\alpha), \dots, x_l(\alpha)$ be continuous complex-valued functions of the parameter α, and for $\alpha \ne 0$, among their values there are no equal ones:$x_i(\alpha) \ne x_j(\alpha)$, and for $\alpha = 0$, among the values of $x_i(0)$ there are groups of equal ones. Let also for $\alpha \ne 0$ the matrix $X(\alpha)$ have the form*

$$X(\alpha) = \begin{pmatrix} 1 & \dots & 1 \\ x_1(\alpha) & \dots & x_l(\alpha) \\ \vdots & & \vdots \\ (x_1(\alpha))^m & \dots & (x_l(\alpha))^m \end{pmatrix},$$

and for each $\alpha \ne 0$ the limit $\lim_{k\to\infty} \|p_k(\alpha)X(\alpha)\| = 0$ holds, where $p_k = (p_0^{[k]}(\alpha), \dots, p_m^{[k]}(\alpha))$. Then for $m \le l$ the sequence has a limit

$$p = \lim_{\alpha\to 0} \lim_{k\to\infty} p_k(\alpha) = 0,$$

and for $m = l+1$ under condition $\|p^{[k]}\| \ge c > 0$ there is a limit $p \ne 0$, moreover, p_j are the coefficients of the polynomial

$$p(x) = p_0 + p_1 x + \dots + p_l x^l,$$

whose roots are all the numbers $x_1(0), \dots, x_l(0)$ taking into account their multiplicity.

Now let's prove the theorem 2.

Proof. We will use the fact that the roots $\lambda_j, j = 1 \dots n$ of the polynomial $f(\lambda)$ coincide with the eigenvalues of the matrix A. This means that the numbers $\xi_j = (1+\lambda_j)/(1-\lambda_j)$ are the eigenvalues of the pencil $A_0 - \xi_j B_0$. Thus, the roots of the polynomial and the spectrum of the pencil are simultaneously divided into two parts by the imaginary axis and the unit circle, respectively. Under the additional assumption that all roots are distinct, it is easy to establish that vectors of the form

$$v_j = (1, \lambda_j, \dots, \lambda_j^{n-1})^T$$

are eigenvectors for both the pencil $A_0 - \xi B_0$ and the matrix A. Consequently, by theorem 3, the equality

$$(A_k - \xi_j^{2^k} B_k) v_j = 0$$

holds. Let $\lambda_1, \dots, \lambda_l$ be the roots of the polynomial $f(\lambda)$ lying in the left half-plane. Then the corresponding eigenvalues $\xi_1, \dots, \xi_l$ lie inside the unit circle and, consequently,

$$\lim_{k\to\infty} |\xi_j^{2^k}| \, \|B_k v_j\| = 0, \;\; j = 1, 2, \dots, l,$$

taking into account that by theorem 3 the norms of $\|B_k\|$ are bounded for all k. From this we obtain the convergence to zero of the sequence $A_k v_j$ as $k \to \infty$. The same is true for $\widetilde{A}_k = \mathrm{ql}(A_k)$

$$\lim_{k\to\infty} \|\widetilde{A}_k v_j\| = 0, \;\; j = 1, 2, \dots, l.$$

From lemma 1 it follows that $\lim_{k\to\infty} \|\widetilde{A}_k^{[j]} = 0$, $1 \le j \le l$, for the upper rows of the matrices $\widetilde{A}_k$. At the same time, for all sufficiently large k, $\|\widetilde{A}_k^{[l+1]} \ge c > 0$. Indeed, as follows from the variational principle ([2], section 7.5)

$$\sigma_{l+1}(A_k) = \sigma_{l+1}(\widetilde{A}_k^*) = \min_{L_{m-1}\subset R_n} \max_{\substack{v\,\in\, L_{m-1},\\ \|v\|\,=\,1}} \|\widetilde{A}_k^* v\| \le$$

$$\le \max_{\substack{v\,=\,\alpha_1 e_1\,+\,\cdots\,+\,\alpha_l e_{l+1},\\ \|v\|\,=\,1}} \|\widetilde{A}_k^* v\|.$$

Here L_{m-1} is a subspace of dimension $m-1$ of the space R_n, e_i are vectors of the standard basis.

Thus, if the norm of the row $\widetilde{A}_k^{[l+1]}$ is not bounded below by a positive constant, then there is no such bound for the singular value $\sigma_{l+1}(A_k)$ either, which contradicts the corollary of theorem 3. Therefore, the second part of assertion 1 is applicable, which means

$$\lim_{k\to\infty} \widetilde{A}_k^{[l+1]} = (b_0, \dots, b_l, 0, \dots, 0).$$

Let us pass to the case of multiple roots. To substantiate the same result, we perturb the coefficients of the polynomial f so that all roots of the perturbed polynomial $\widetilde{f}$ are different $\widetilde{\lambda}_j \neq \widetilde{\lambda}_i$, $i \neq j$, but the number of roots in the right and left half-planes does not change. We connect the corresponding points λ_j and $\widetilde{\lambda}_j$ by a parametrized continuous curve $z_j(\alpha)$ so that $z_j(1) = \widetilde{\lambda}_j$, $z_j(0) = \lambda_j$, $z_j(\alpha) \neq z_i(\alpha)$ for $i \neq j$ and $\alpha \neq 0$. Thus, each α will correspond to a sequence of penciles $A_k(\alpha) - \xi B_k(\alpha)$. Note that the result of the qr-transformation, which is used to construct the sequence of penciles, continuously depends on the elements of the original matrix up to the sign of the rows.

Matrices $T(\alpha), S(\alpha), \Lambda(\alpha), M(\alpha)$ of the canonical decomposition of the pencil $A_0(\alpha) - \xi B_0(\alpha)$ (see theorem 4) can also be chosen to depend continuously on the parameter $\alpha \in [0, 1]$. Since continuous functions on a compact set attain a maximum and a minimum, then

$$0 < c \le \sigma_j(T(\alpha)), \sigma_j(S(\alpha)) \le C < \infty.$$

This means that repeating the arguments given above, we obtain that for all sufficiently large k $\|\widetilde{A}_k^{[l+1]} \geq c > 0$. Thus, lemma 3 is applicable, which completes the proof of the theorem in the case of multiple roots for the matrix A_k and the factor $g(\lambda)$.

For the matrix B_k and the factor $h(\lambda)$ the proof is similar. □

3.3. Example. Consider the Chebyshev polynomial $f(\lambda) = 10-2\lambda-\lambda^2+2\lambda^3+\lambda^4 = g(\lambda)h(\lambda)$, where

$$g(\lambda) = 2 - 2\lambda + \lambda^2, \qquad h(\lambda) = 5 + 4\lambda + \lambda^2.$$

Construct the associated matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & 2 & 1 & -2 \end{bmatrix}.$$

After 6 iterations, the matrices $\widetilde{A}_k$, $\widetilde{B}_k$ have the form

$$\widetilde{A}_k = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1.2599 & -1.0079 & -0.2520 & 0 \\ -2.1567 & -0.2876 & 0.7189 & 0.2876 \end{bmatrix},$$

$$\widetilde{B}_k = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1.0887 & -1.0887 & 0.5443 & 0 \\ -3.2660 & 1.0887 & 0.5443 & -1.0887 \end{bmatrix}.$$

Let us determine the relative errors of the calculated coefficients δ_g and δ_h

$$\delta_g = \frac{\|(g_0, g_1, g_2) - (2, -2, 1)\|}{\|(2, -2, 1)\|} = 1.8957 \cdot 10^{-15},$$

$$\delta_h = \frac{\|(h_0, h_1, h_2) - (5, 4, 1)\|}{\|(5, 4, 1)\|} = 2.6469 \cdot 10^{-15}.$$

As can be seen, the accuracy of the result is quite high.

We apply the algorithm to several Chebyshev polynomials $T_k(\lambda)$ of degree $k = 4.5, \ldots 10$ and obtain pairs of vectors g_k, h_k whose components are the coefficients of the factors $g_k(\lambda), h_k(\lambda)$. Let $t_k(\lambda) = g_k(\lambda) \cdot h_k(\lambda)$ and t_k is the vector of its coefficients. An impression of the accuracy of the decomposition result is given by the value

$$\delta_k = \frac{\|T_k - t_k\|}{\|T_k\|}.$$

k	4	5	6	7	8	9	10
$-\log_{10}\delta$	15.09	-	14.68	-	13.84	-	11.82
$\log_{10}\omega$	1.13	>16	2.34	>16	3.66	>16	5.04

Note that if k is odd, then the separation of roots is impossible, since one of the roots is zero. This situation is diagnosed using the dichotomy criterion ω, which exceeds the specified upper limit of admissible values 10^{16}. With the growth of even powers of k, a gradual increase in the error and the dichotomy criterion is observed.

4 Appendix

4.1. An algorithm for dichotomy with respect to unit circle. There are several variants of the unit circle dichotomy algorithms [5]-[7] . The basic structure of each of them is a cycle of sequentially computing pairs of matrices A_k, B_k using the qr-decomposition of the composite matrix (10). At that, the convergence conditions can be formulated in various ways, including a priori estimates for the number of iterations required for the cycle to converge under the condition $\|H\| \le \omega_{max}$ (e.g., [8], Theorem 5) . However, it should be taken into account that such estimates are usually very high. As a rule, a few iterations are sufficient for convergence.

Algorithm for dichotomy of matrix spectrum with respect to unit circle

Given: matrix pencil $A_0 - \lambda B_0$, ε_{it} – required accuracy of iteration process, ω_{max}, μ_{max} – maximum values of dichotomy criterion and matrix condition number.
If $\mathrm{cond}(\mathrm{A_0} - \mathrm{B_0}) > \mu_{\max}$,
then dichotomy is impossible, end of calculations.
else

$$H_0 = (A_0 - B_0)^{-1}(A_0A_0^* + B_0B_0^*)(A_0^* - B_0^*)^{-1}$$

while $\|H_k - H_{k-1}\| > \varepsilon_{it}\|H_k\|$
If $\|H_k\| \ge \omega_{max}$ or $\mathrm{cond}(\mathrm{A_k} + \mathrm{B_k}) > \mu_{\max}$,
then dichotomy is impossible, end of calculations.
else

$$V_{k+1} = (A_k + B_k)^{-1}A_k, \quad U_{k+1} = I - V_{k+1}$$

$$H_{k+1} = U_{k+1}H_kU_{k+1}^* + V_{k+1}H_kV_{k+1}^*$$

$$qr\left(\begin{bmatrix} -B_k & A_k & 0 \\ A_k & 0 & -B_k \end{bmatrix}\right) = \begin{bmatrix} * & * & * \\ 0 & A_{k+1} & -B_{k+1} \end{bmatrix}$$

$$P_k = -(A_{k+1} - B_{k+1})^{-1}B_{k+1}$$

End of cycle
Result: dichotomy criterion $\omega = \|H_k\|$, projector $P = P_k$.

If $\|H^{(k)}\|$, calculated at step k, exceeds the specified value ω_{max}, then the circle intersects regions on the complex plane within which $\sigma_{\min}(A_0 - \lambda B_0) < \varepsilon$ for sufficiently small $\varepsilon > 0$. This is how the situation is diagnosed when it is impossible to separate the spectrum reliably enough.

4.2. Theorems about canonical decomposition.

Theorem 4. *Let $A - \xi B$ have no eigenvalues on the unit circle, then there exist non-singular $n \times n$ matrices T and S, as well as matrices Λ of size $l \times l$ and M of size $m \times m$, all of whose eigenvalues lie inside the unit circle, such that the decomposition*

$$A = T \begin{pmatrix} \Lambda & 0 \\ 0 & I_m \end{pmatrix} S, \qquad B = T \begin{pmatrix} I_l & 0 \\ 0 & M \end{pmatrix} S.$$

Here I_l and I_m are the identity matrices of size $l \times l$ and $m \times m$, respectively.

For proof, see [2] section 10.3.

Now let's prove the theorem 3.

Proof. Consider auxiliary matrix pencils:$A'_k - \xi B'_k = A_k S^{-1} - \xi B_k S^{-1}$. By construction, for them, as well as for $A_k - \xi B_k$, the relations

$$Q^*_{k+1} \begin{bmatrix} -B'_k & 0 & A'_k \\ A'_k & -B'_k & 0 \end{bmatrix} = \begin{bmatrix} * & * & * \\ 0 & -B'_{k+1} & A'_{k+1} \end{bmatrix} \tag{11}$$

We will show that the pencil $A_k - \xi B_k$ has a canonical decomposition

$$A'_k = T_k \begin{pmatrix} \Lambda^{2^k} & 0 \\ 0 & I \end{pmatrix}, \quad B'_k = T_k \begin{pmatrix} I & 0 \\ 0 & M^{2^k} \end{pmatrix}. \tag{12}$$

Note that from the theorem 4 it follows that for k=0 there is a basis for induction. Let the equalities (12) hold for some value of the index $k \geq 0$. Represent the matrices T_k and Q_{k+1} in cellular form:

$$T_k = \begin{pmatrix} T^{(k)}_{11} & T^{(k)}_{12} \\ T^{(k)}_{21} & T^{(k)}_{22} \end{pmatrix}, \quad Q^*_k = \begin{pmatrix} * & * \\ \Psi_k & \Phi_k \end{pmatrix},$$

$$\Psi_k = \begin{pmatrix} \Psi^{(k)}_{11} & \Psi^{(k)}_{12} \\ \Psi^{(k)}_{21} & \Psi^{(k)}_{22} \end{pmatrix}, \quad \Phi_k = \begin{pmatrix} \Phi^{(k)}_{11} & \Phi^{(k)}_{12} \\ \Phi^{(k)}_{21} & \Phi^{(k)}_{22} \end{pmatrix}.$$

Let us introduce additional notations: $X_K = \Psi_{k+1} T_k$ and $Y_k = \Phi_{k+1} T_k$. Note that (11) implies the equality $\Psi_{k+1} A'_k + \Phi_{k+1} B'_k = 0$. Taking this into account, it is easy to verify the validity of the representation

$$X_k = T_{k+1} \begin{pmatrix} I & 0 \\ 0 & M^{2^k} \end{pmatrix}, Y_k = T_{k+1} \begin{pmatrix} \Lambda^{2^k} & 0 \\ 0 & I \end{pmatrix}. \tag{13}$$

According to (11) and the notations proposed above, the following equalities hold:

$$A'_{k+1} = \Psi_{k+1} A'_k = X_k \begin{pmatrix} \Lambda^{2^k} & 0 \\ 0 & I \end{pmatrix}, \quad B'_{k+1} = \Phi_{k+1} B'_k = Y_k \begin{pmatrix} I & 0 \\ 0 & M^{2^k} \end{pmatrix} \tag{14}$$

By combining (13) and (14), we prove the first part of the theorem.

In order to prove the theorem completely, it remains to estimate the singular values of the matrix T_k. By the definition of the matrices X_k, Y_k and T_k the equality

$$T_{k+1}\begin{bmatrix} I & 0 & \Lambda^{2^{k+1}} & 0 \\ 0 & M^{2^{k+1}} & 0 & I \end{bmatrix} = (X_k \vdots Y_k) = (\Phi_{k+1} \vdots \Psi_{k+1}) \begin{pmatrix} T_k & \\ & T_k \end{pmatrix}$$

holds. We use the following inequalities for the singular values of the matrix product ([2], Section 7.5):

$$\sigma_j(A)\sigma_1(B) \le \sigma_j(AB) \le \sigma_j(A)\sigma_n(B).$$

We obtain

$$\sigma_j(T_{k+1})\sigma_1\left(\begin{bmatrix} I & 0 & \Lambda^{2^{k+1}} & 0 \\ 0 & M^{2^{k+1}} & 0 & I \end{bmatrix}\right) \le \sigma_j([X_k \vdots Y_k]) \le$$

$$\le \sigma_j([\Phi_{k+1} \vdots \Psi_{k1}])\sigma_n\left(\begin{bmatrix} T_k & \\ & T_k \end{bmatrix}\right)$$

in one direction and

$$\sigma_j(T_{k+1})\sigma_n\left(\begin{bmatrix} I & 0 & \Lambda^{2^{k+1}} & 0 \\ 0 & M^{2^{k+1}} & 0 & I \end{bmatrix}\right) \ge \sigma_j([X_k \vdots Y_k]) \ge$$

$$\ge \sigma_j([\Phi_{k+1} \vdots \Psi_{k1}])\sigma_1\left(\begin{bmatrix} T_k & \\ & T_k \end{bmatrix}\right)$$

in another. Note that

$$1 \le \sigma_j\left(\begin{bmatrix} I & 0 & \Lambda^{2^{k+1}} & 0 \\ 0 & M^{2^{k+1}} & 0 & I \end{bmatrix}\right) \le 1 + \max\left\{\left\|\Lambda^{2^k}\right\|, \left\|M^{2^k}\right\|\right\}$$

Hence

$$\sigma_1(T_{k+1})\left(1 + \max\left\{\left\|\Lambda^{2^k}\right\|, \left\|M^{2^k}\right\|\right\}\right)^{-1} \le \sigma_j(T_k) \le \sigma_n(T_{k-1})$$

Thus we finally get

$$\sigma_1(T)\prod_{i=0}^{k}\left(1 + \max\left\{\left\|\Lambda^{2^i}\right\|, \left\|M^{2^i}\right\|\right\}\right)^{-1} \le \sigma_j(T_k) \le \sigma_n(T),$$

which is what was required to be proved. □

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