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ON INCREASING THE ACCURACY OF THE GODUNOV SCHEME FOR GAS-DYNAMIC AND ELASTOPLASTIC FLOWS

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Dedicated to the memory of Sergey Godunov

Abstract: The contribution of Sergei Konstantinovich Godunov to the development of numerical methods is difficult to overestimate. One of the authors, Abuziarov M.H., participated in the work of the international symposium "The Godunov Method in Gas Dynamics"at the University of Michigan (An Arbor USA) in May 1997, organized by NASA USA in honor of Godunov. At this symposium, Sergei Konstantinovich was presented as the most outstanding applied mathematician of the 20th century, and a special tour of NASA laboratories in the USA was organized for him. The scheme originally proposed by Godunov for solving the equations of gas dynamics has also found wide application for solving elastic-plastic problems of continuum mechanics. This finite-volume flow scheme of the predictor-corrector type is based on solving the Riemann problem at the predictor step at the cell boundaries under the assumption of a piece-wise constant distribution of parameters in these cells. The main advantages of the scheme, such as monotonicity, the possibility of identifying impact and contact discontinuities, the use of Lagrangian and Eulerian approaches, and the simplicity of implementing boundary conditions are the consequences of solving this problem. At the same

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time, solving the Riemann problem using a piecewise constant distribution of parameters in cells is the source of the main disadvantage of the scheme - significant scheme viscosity, while the corrector step of the scheme is sufficient for the second-order of approximation. In this paper, the authors proceed from the linear distribution of flow parameters between the centers of neighboring cells, and from the analysis of the differential approximation of the scheme, by the appropriate choice of these parameters for solving the Riemann problem, they increase the order of approximation of the scheme to the second in the region of smooth solutions while maintaining monotonicity on discontinuous ones. The appropriate choice of these parameters increases the order of approximation of the scheme to the second in space and time on a moving non-uniform grid for both the Lagrangian and Eulerian cases on a compact stencil, for both gas-dynamic and elastic-plastic flows. Monotonicity near discontinuous solutions is ensured by switching to the predictor step of the first-order accuracy scheme. The coordinates of the interpolation points of the flow parameters have an obvious physical meaning - these are the boundaries of the dependence region of the Riemann problem solution for the moving coordinate of the face center at half the time integration step. This scheme modification is practically the same for both gas-dynamic and elastic-plastic flows. In contrast to gas-dynamic problems, for elastic-plastic flows, the corresponding Riemann invariants are interpolated at the boundaries of the dependence regions. The quality of the scheme is illustrated by test problems.

Keywords: Godunov scheme, high accuracy, differential approximation, three-dimensionality, gas dynamics, elastic-plastic flows, finite volume method.

1 INTRODUCTION

A difference scheme based on the exact solution of the Riemann problem (RP) was proposed by S.K. Godunov [1], [2] for the numerical solution of multidimensional gas-fluid dynamics problems (CFD). It became the basis for creating of a whole family of schemes called as finite volume methods (FVM), including those for modeling nonlinear elastic-plastic processes in continuum mechanics (computational solid dynamics - CSD). These methods are based on the use of the RP solution, which determines the main advantages of this class of schemes: the ability to isolate and track shock waves and contact discontinuities, the use of Lagrangian and Eulerian variables, and monotonicity without introducing artificial viscosity. The solution of the same RP, when using piecewise constant parameter values in adjacent cells, is also the cause of the main drawback of the original Godunov scheme - significant scheme viscosity. Accordingly, the aim of numerous modifications of the Godunov scheme [3]-[10] and others was to reduce this viscosity. Since the corrector step of the Godunov scheme is sufficient for the second-order of approximation, the predictor step was modified – the initial data for the RP solution were changed. Several variants were also proposed where the corrector step was additionally corrected, for example, the work [9]. The Godunov scheme assumes a piecewise constant distribution of parameters in the cells, and the calculation of the RP is performed at the boundary between these discontinuous parameters. Almost all modifications of the Godunov scheme also assume a discontinuity at the boundary between the cells, only the values of the parameters to the left and right of the discontinuity change, determined by different laws of parameter distribution inside the cells (linear [3], [4], parabolic [6]). It is assumed that the discontinuity of the parameters is at this boundary and a one-dimensional RP solution is used. Accordingly, as a consequence: 1-the difference stencil increases from the original Godunov scheme 3x3x3 to 5x5x5, which complicates the algorithm, especially in the case of unstructured grids, and leads to the loss of the hyperbolicity property of the difference equations; 2-the problem of constructing a spatial distribution in cells in 2D and 3D cases arises; the stability conditions become more stringent; different algorithms are required for sub- and supersonic flows, as well as for Lagrangian and Eulerian variables. In this paper, an approach is proposed for constructing Godunov-type difference schemes (development of [10]-[12]) based on the analysis of the parametric differential approximation of the scheme for linearized equations, which makes it possible to obtain parameters for solving RP, providing the possibility of introducing adjustable scheme viscosity with the possibility of continuous transition from a scheme with zero viscosity (second-order) to a Godunov scheme of the first-order of accuracy. A linear distribution of flow parameters between the centers of neighboring cells is assumed, and the choice of these parameters for solving RP, provides the second-order of approximation in the region of smooth solutions and monotonicity on discontinuous ones. To implement this approach, a parametric expression for the first differential approximation for linearized equations for non-uniform difference grids is obtained, where the coordinates of the interpolation points of the flow parameters vary. The choice of the appropriate coordinates of these points makes it possible to regulate the viscosity of the scheme for linearized equations; in fact, the dependence of the RP solution on time and space is introduced. The RP solution obtained in this way increases the order of approximation of the scheme on a compact 3x3x3 template to the second in space and time on orthogonal non-uniform moving grids for 3D linearized equations. This approach has an obvious physical meaning - convergence of the influence areas of the difference and differential problems. In this case, only the predictor step of the scheme is changed.

2 SYSTEM OF EQUATIONS FOR NUMERICAL MODELING OF GAS-DYNAMIC AND ELASTOPLASTIC FLOWS

For modeling, integral equations are used in the form of the laws of conservation of mass, momentum, and energy for an arbitrary moving volume, describing the deformation of a continuous medium in the approximation of a compressible elastic-plastic solid model [13], [14]. These equations can be used to describe both smooth and discontinuous solutions. In a Cartesian coordinate system, the corresponding differential equations have the following form [14]:

$$\rho_{,t} + (\rho u_i)_{,x_i} = 0 \tag{1.1}$$

$$(\rho u_i)_{,t} + (\rho u_i u_j - \sigma_{ij})_{,x_j} = 0 \tag{1.2}$$

$$e_{,t} + (e u_j - u_i \sigma_{ij})_{,x_j} = 0 \tag{1.3}$$

$$DS_{ij}/Dt + \lambda_i S_{ij} = 2\mu e_{ij} \tag{1.4}$$

$$\varepsilon = \varepsilon(p, \rho), \tag{1.5}$$

Where t is time, $x_i, i = 1, 2, 3$ are the spatial Eulerian coordinates, u_i are the components of the velocity vector along the axes x_i respectively, ρ is the density, $e = \rho(\varepsilon + u_i u_i/2)$ is the total energy per unit volume, ε is the internal energy of a unit mass given by the equation of state (1.5), σ_{ij} is the stress tensor, which is represented as spherical and deviatoric parts $\sigma_{ij} = -p\delta_{ij} + S_{ij}, p = -\sigma_{ii}/3$, e_{ij} is the strain rate tensor deviator $e_{ij} = \varepsilon_{ij} - 1/3\varepsilon_{kk}\delta_{ij}$, where $\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$, and the indices after the decimal point refer to the corresponding differentiation (in the time or the corresponding direction), δ_{ij} is the Kronecker symbol. The symbol D/Dt indicates the Jaumann derivative, taking into account the stress tensor rotation in Eulerian variables (hypoelastic model). $DS_{ij}/Dt = S_{ij,t} + u_k \partial S_{ij} \partial x_k - S_{ik}\omega_{jk} - S_{jk}\omega_{ik}$, where $\omega_{ij} = (u_{i,j} - u_{j,i})/2$, and μ is the material's shear modulus. The von Mises yield condition is used as a criterion for the transition from an elastic to a plastic state $S_{ij}S_{ij} = 2/3\sigma_S^2$, and σ_S is the yield stress in the uniaxial tension. The parameter λ must remain positive during plastic deformation when the von Mises yield condition is satisfied $\lambda_t = 3/2 S_{ij}S_{ij}/\sigma_S^2$. Plastic flow is described by maintaining the value of the deviator on the yield surface [13]. In the absence of shear stresses, system (1.1)-(1.5) obviously transforms into the Euler equations of motion of an ideal compressible liquid or gas.

3 MODIFICATION OF THE RIEMANN PROBLEM FOR LINEARIZED EQUATIONS BASED ON THE ANALYSIS OF DIFFERENTIAL APPROXIMATION

3.1. The Riemann problem for the Godunov scheme of increased accuracy for one-dimensional Euler equations.

Let's denote the components of the velocity vector along the axes as in [2] , is the sound speed. After linearization near the vector for the one-dimensional case, we have [2]:

$$u_t + u_0 u_x + p_x/\rho_0 = 0 \quad (2.1)$$

$$p_t + u_0 p_x + \rho_0 c_0^2 u_x = 0 \quad (2.2)$$

$$d\rho/dt = dp/dt/c_0^2 \quad (2.3)$$

For subsequent obtaining and analysis of the parametric differential approximation, it is sufficient to consider equations (2.1), (2.2). The goal is to introduce into the Godunov scheme for equations (2.1) and (2.2) at the predictor stage an explicit dependence on the choice of flow parameters used to solve RP, to construct the first differential approximation of this scheme depending on these parameters and to modify it to obtain the required approximation. Let's assume a linear distribution of flow parameters between the cell centers and the absence of a discontinuity at the cell boundary. From here on, parameters with integer index values will be related to the cell centers, the integer index at the bottom will mean the parameter value on the lower time layer, at the top - accordingly on the upper time layer, half-integer indices will refer to the coordinates of the cell boundaries and the "breakup" (at the intermediate time layer) values of the parameters at these boundaries. In Cartesian coordinates for a cell centered at the point (x_i, y_j, z_k) and with boundaries along the axis x $(x_{i-1/2}, y_j, z_k)$, $(x_{i+1/2}, y_j, z_k)$ for the time layer t_n this will be the following distribution of flow parameters U, where U denotes parameters (p, ρ, u, v, w).

At $x_{i-1} \le x \le x_i$

$U(x, y_j, z_k) = U(x_{i-1}, y_j, z_k) + (U(x_i, y_j, z_k) - U(x_{i-1}, y_j, z_k))/(x_i - x_{i-1})(x - x_{i-1})$ and

at $x_i \le x \le x_{i+1}$

$U(x, y_j, z_k) = U(x_i, y_j, z_k) + (U(x_{i+1}, y_j, z_k) - U(x_i, y_j, z_k))/(x_{i+1} - x_i)(x - x_i)$,(fig.1)

Grid step along the axis x $h_i = x_{i+1/2} - x_{i-1/2}$. The same applies to the directions y, z. We will solve the Riemann problem for a cell-centered at a point (x_i, y_j, z_k) at the border $(x_{i-1/2}, y_j, z_k)$ between the parameters $U(x^+_{i-1}, y_j, z_k)$ and $U(x^-_i, y_j, z_k)$, and at the border $(x_{i+1/2}, y_j, z_k)$ between the parameters $U(x^+_i, y_j, z_k)$ and $U(x^-_{i+1}, y_j, z_k)$. The coordinates of the points $(x^+_{i-1}, y_j, z_k), (x^-_i, y_j, z_k)$ and $(x^+_i, y_j, z_k), (x^-_{i+1}, y_j, z_k)$ will be changed. Similarly for other faces, the goal is to obtain a differential approximation depending on these coordinates.

For 1D equations (2.1) and (2.2) Godunov's scheme looks like this

$$(u^i - u_i)/\Delta t + u_0(u_{i+1/2} - u_{i-1/2})/\Delta x + (p_{i+1/2} - p_{i-1/2})/\rho_0/\Delta x = 0$$

$$(p^i - p_i)/\Delta t + u_0(p_{i+1/2} - p_{i-1/2})/\Delta x + \rho_0 c_0^2(u_{i+1/2} - u_{i-1/2})/\Delta x = 0$$

For the first-order scheme (definition of the values $u_{i-1/2}, p_{i-1/2}, u_{i+1/2}, p_{i+1/2}$) RP is solved for the parameters at the cell centers, and its solution for the subsonic case has the fol-lowing form [2]:

$$u_{i-1/2} = (u_{i-1} + u_i)/2 + (p_{i-1} - p_i)/(2\rho_0 c_0);$$

$$p_{i-1/2} = (p_{i-1} + p_i)/2 + \rho_0 c_0(u_{i-1} - u_i)/2;$$

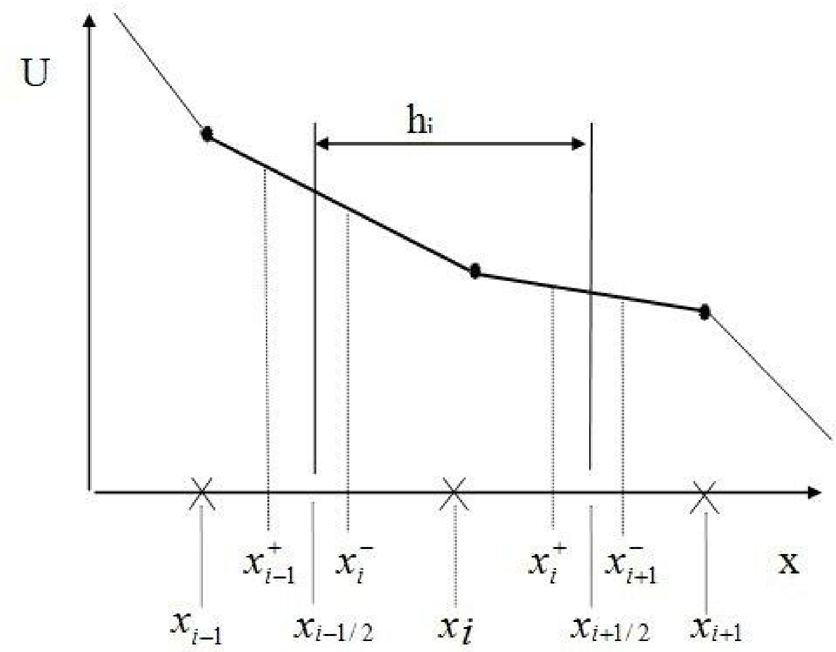


FIG 1. linear distribution of flow parameters between the centers of adjacent cells.

$$u_{i+1/2} = (u_i + u_{i+1})/2 + (p_i - p_{i+1})/(2\rho_0 c_0);$$
$$p_{i+1/2} = (p_i + p_{i+1})/2 + \rho_0 c_0 (u_i - u_{i+1})/2;$$

For the supersonic case, these relations are given in [2].

For the high-precision scheme, the parameters for the RP solution are determined at the points $x_{i-1}^+, x_i^-, x_i^+, x_{i+1}^-$ by linear interpolation from the cell centers (here and below x_{i-1}^+, x_i^- coordinates relative to the boundary $x_{i-1/2}$, and x_i^+, x_{i+1}^- relative to the boundary $x_{i+1/2}$, $u_{i-1}^+, u_i^-, p_{i-1}^+, p_i^-, u_i^+$, $u_{i+1}^-, p_i^+, p_{i+1}^-$ (respectively, the values at these points), and the solution to this problem, regardless of the flow regime, has the following form (Fig. 1):

$$u_{i-1/2} = (u_{i-1}^+ + u_i^-)/2 + (p_{i-1}^+ - p_i^-)/(2\rho_0 c_0);$$
$$p_{i-1/2} = (p_{i-1}^+ + p_i^-)/2 + \rho_0 c_0 (u_{i-1}^+ - u_i^-)/2;$$
$$u_{i+1/2} = (u_i^+ + u_{i+1}^-)/2 + (p_i^+ - p_{i+1}^-)/(2\rho_0 c_0);$$
$$p_{i+1/2} = (p_i^+ + p_{i+1}^-)/2 + \rho_0 c_0 (u_i^+ - u_{i+1}^-)/2;$$

For this solution, we obtain the following first differential approximation (the derivation is given in [14]):

$$u_t + u_0 u_x A_1 + p_x/\rho_0 B_1 = -u_{xx} A_2/2 - p_{xx}/(\rho_0 c_0) B_2/2 + \bar{o}(h_i^2, \Delta t^2, \Delta t h_i)$$

$$p_t + u_0 p_x A_1 + \rho_0 c_0^2 u_x B_1 = -p_{xx} A_2/2 - \rho_0 c_0 u_{xx} B_2/2 + \bar{o}(h_i^2, \Delta t^2, \Delta t h_i)$$
$$A_1 = (x_i^+ + x_{i+1}^- - x_{i-1}^+ - x_{i-1}^- + c_0/u_0(x_i^+ - x_{i+1}^- + x_i^- - x_{i-1}^+))/(2h_i)$$
$$B_1 = (x_i^+ + x_{i+1}^- - x_{i-1}^+ - x_{i-1}^- + u_0/c_0(x_i^+ - x_{i+1}^- + x_i^- - x_{i-1}^+))/(2h_i)$$

$$A_2 = \Delta t(c_0^2 + u_0^2) + u_0/(2h_i)((h_{i-1} + h_i)(x_{i-1}^+ + x_i^-) + (h_i + h_{i+1})(x_i^+ + x_{i+1}^-))/2 + \\ + c_0/(2h_i)((h_{i-1} + h_i)(x_{i-1}^+ - x_i^-) + (h_i + h_{i+1})(x_i^+ - x_{i+1}^-))/2 \quad (2.4)$$

$$B_2 = 2u_0 c_0 \Delta t + u_0/(2h_i)((h_{i-1} + h_i)(x_{i-1}^+ - x_i^-) + (h_i + h_{i+1})(x_i^+ - x_{i+1}^-))/2 + \\ + c_0/(2h_i)((h_{i-1} + h_i)(x_{i-1}^+ + x_i^-) + (h_i + h_{i+1})(x_i^+ + x_{i+1}^-))/2 \quad (2.5)$$

where $\Delta t = t_{n+1} - t_n$; $h_i = x_{i+1/2} - x_{i-1/2}$; $h_{i-1} = x_{i-1/2} - x_{i-3/2}$; $h_{i+1} = x_{i+3/2} - x_{i+1/2}$. For the first- order Godunov scheme [2] these coefficients are $A_1 = 1$; $B_1 = 1$; $A_2 = \Delta t(c_0^2 + u_0^2) - h_i c_0$; $B2 = 2\Delta t u_0 c_0 - h_i u_0$. It is obvious that the second-order for this scheme is possible only if $u_0 = 0$, $\Delta t c_0 = h_i$ (acoustics, the Courant number is 1). The modified scheme at $A_1 = 1$; $B_1 = 1$ ($x_{i-1}^+ = x_i^+$;$x_i^- = x_{i+1}^-$) approximates the linearized equations with the first-order of accuracy. That is, for the first-order of approximation, the left interpolation points x_{i-1}^+, x_i^+(x_{i-1}^+ to the left of the boundary $x_{i-1/2}$ and x_i^+ to the left of the boundary $x_{i+1/2}$) must be at the same distance from the corresponding boundaries. Similarly, the same distances for the right x_i^-, x_{i+1}^-. Obviously, this condition will not be satisfied for the first-order Godunov scheme on a non-uniform grid, and it will not even have a first-order approximation on such a grid. In order to develop the modified scheme with a second-order approximation, it is necessary to add the following conditions: $A_2 = 0$; $B_2 = 0$. Accordingly, equations (2.4) and (2.5) yield:

$$x_{i-1}^+ = (\Delta t(-u_0 - c_0)/2 + (h_{i+1} - h_{i-1})/8)/(1 + (h_{i+1} + h_{i-1} - 2h_i)/(4h_i)); x_i^+ = x_{i-1}^+$$
$$x_i^- = (\Delta t(-u_0 + c_0)/2 + (h_{i+1} - h_{i-1})/8)/(1 + (h_{i+1} + h_{i-1} - 2h_i)/(4h_i)); x_{i+1}^- = x_i^- \quad (2.6)$$

Formulas (2.6) define the coordinates of the interpolation points. The interpolation of the parameters from the cell centers to these points with subsequent calculation in the standard RP manner provides the second-order approximation of the 1D linearized Euler equations. In this case, the densities at these points are determined from equation (2.3). The RP solution for these parameters will depend on the integration step over time and the difference grid. For a uniform grid or a grid changing according to the law of arithmetic progression with a step $h_i - h_{i-1} = h_{i+1} - h_i = \Delta$, we get the following:

$$x_{i-1}^+ = (\Delta t(-u_0 - c_0)/2 + \Delta/4); x_i^+ = x_{i-1}^+; x_i^- = (\Delta t(-u_0 + c_0)/2 + \Delta/4); x_{i+1}^- = x_i^- \quad (2.7)$$

These coordinates have an obvious physical meaning. They limit the domain of influence on the RP solution for the face of the integrated cell at the time instant $\Delta t/2$, i.e. at $\Delta t/2$ when the flows through the corresponding face are determined from the RP solution, disturbances can only be from the domain limited by the extreme characteristics arriving at this face. The size of the domain of influence for all cases is $c_0 \Delta t/2$ and coincides with the domain of influence of the differential problem. Figure 2 shows a geometric interpretation of (2.7) for the cell face $x_{i-1/2}$

(special cases): 1-acoustic case - the domain of influence $c_0\Delta t$ and, accordingly, the interpolation points are symmetrical with respect to the boundary; 2- subsonic case, upstream displacement of $c_0\Delta t$; 3- supersonic case, upstream displacement by one cell of $c_0\Delta t$; 4- non-uniform grid, displacement of $c_0\Delta t$ toward a larger cell; 5- moving edge with velocity W, the interpolation points and the domain of influence $c_0\Delta t$ are determined with respect to the position of the face at the moment $c_0\Delta t/2$. In the first-order accuracy scheme, this domain of influence does not depend on time and is equal to the cell size $h_i \geq c_0\Delta t$.

3.2. Modification of the Riemann problem for three-dimensional linearized Euler equations.

In the Godunov scheme [2] for 2D and 3D cases, splitting by spatial variables is used. For each face in the normal direction, a one-dimensional Riemann problem is solved for pressures, densities, and velocities normal to the face. The tangential velocity components are selected from the centers of the corresponding donor cells. The use of an algorithm that increases the accuracy of the scheme to the second-order in the one-dimensional case is insufficient for 2D and 3D. From the analysis of the parametric equations obtained for the spatial equations, an additional correction is needed before solving the RP to increase the accuracy in space and time. Before interpolation, it is necessary to additionally take into account the influence of the tangential (y and z) components of the equations on the interpolated parameters, including the y and z components of the velocities. In what follows, the RP solution for CFD will be called the solution with y and z components of the velocities. Let's denote these refined parameters by symbols with asterisks $U^* = (p^*, \rho^*, u^*, v^*, w^*)$. For the Riemann problem at the boundary $i-1/2, j, k$ (the cell i, j, k is integrated), these refined parameters are $p^*_{i-1}, u^*_{i-1}, v^*_{i-1}, w^*_{i-1}, p^*_i, u^*_i, v^*_i, w^*_i$ where

$$u^*_i = u_1 - \Delta t/2(v_0 u_y + w_0 u_z);$$
$$p^*_i = p_1 - \Delta t/2(v_0 u_y + w_0 p_z + \rho_0 c_0^2(v_y + w_z));$$
$$v^*_i = v_1 - \Delta t/2(v_0 v_y + w_0 v_z + p_y/\rho_0);$$
$$w^*_i = w_1 - \Delta t/2(v_0 w_y + w_0 w_z + p_z/\rho_0) \quad (2.8)$$

Similarly, the refined parameters $p^*_{i-1}, u^*_{i-1}, v^*_{i-1}, w^*_{i-1}$ for the cell $i-1, j, k$ are determined. The derivatives with respect to y and z on the right-hand sides of (2.8) for the cells $i-1, j, k$ and i, j, k are calculated with the second-order of accuracy (by three points). For the y-axis, these are cells centered at the points $(i, j-1, k)$, (i, j, k), $(i, j+1, k)$, and for the z-axis respectively $(i, j, k-1)$, $(i, j, k1)$, $(i, j, k+1)$. Next, we perform interpolation of values $p^*(x_{i-1}, y_j, z_k)$, $u^*(x_{i-1}, y_j, z_k)$, $p^*(x_i, y_j, z_k)$, $u^*(x_i, y_j, z_k)$ at the points x^+_{i-1} and x^-_i. The densities at these points are determined from equation (2.3). Next, for these parameters, the one-dimensional Riemann problem is solved [2], from which we obtain $p_{i-1/2,j,k}$,$u_{i-1/2,j,k}$,$\rho_{i-1/2,j,k}$. The tangential components of the velocities are determined as follows:

$$v_{i-1/2,j,k} = (v^*_{i-1,j,k} + v^*_{i,j,k})/2 - u_0\Delta t/2(v^*_{i,j,k} - v^*_{i-1,j,k})/h_x$$

$$w_{i-1/2,j,k} = (w^*_{i-1,j,k} + w^*_{i,j,k})/2 - u_0\Delta t/2(w^*_{i,j,k} - w^*_{i-1,j,k})/h_x$$

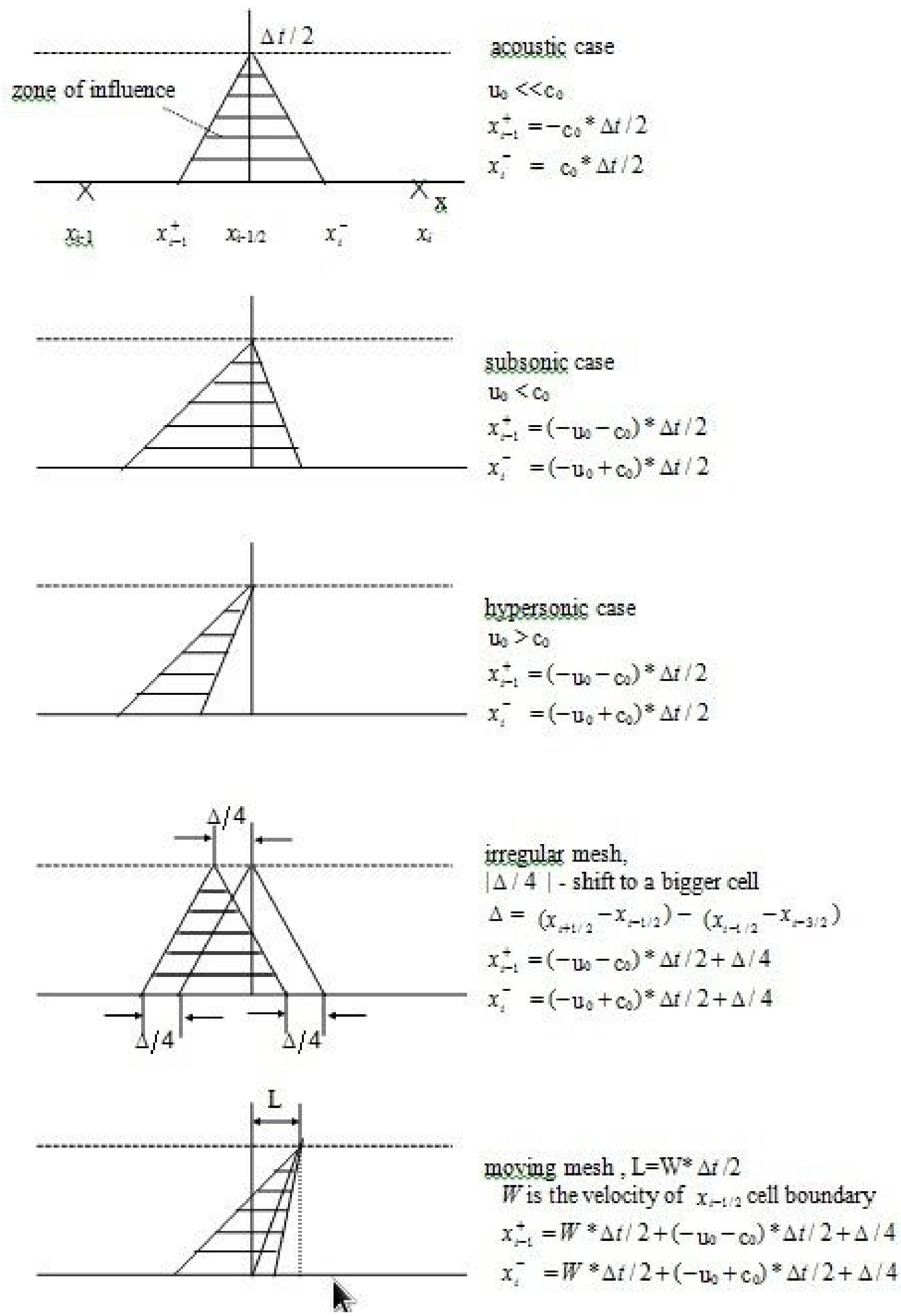


Fig 2. Zone of influence on the Riemann solution at the cell boundary at $\Delta t/2$.2

In fact, they are determined from a solution of the transport equations in the x direction for the equations of motion in the y and z directions (interpolation to a point with coordinates $x_{i-1/2,j,k} - u_0 \Delta t/2$) taking into account the influence of the gradients in the x and y directions. The parameters obtained in this way are used in further calculation of flows and integration according to the standard Godunov scheme and provide an approximation of 3D linearized equations with the second-order of accuracy in space and time. This approach is valid for both subsonic and supersonic flow regimes.

3.3. Modification of the Riemann problem for a deformable solid.

To simulate the dynamics of a compressible elastic-plastic medium, the method of splitting elastic-plastic equations proposed by V.N. Kukudzhanov [14] is used. This method allows to significantly simplify the process of calculating elastic-plastic flows and reduces the calculation of plastic behavior to the correction of the elastic solution while maintaining the second- order of approximation. Within the framework of the Godunov scheme, this means that it is sufficient to integrate the elastic equations - the elastic Riemann problem is solved, the calculation and integration of elastic flows and components of the stress tensor deviator with subsequent correction of elastic stresses at the corrector stage depending on the plasticity model are used [14]. For the case of ideal plasticity, the correction of the elastic solution coincides with that proposed by Wilkins [16] $S_{ij} = S_{ij}/\sqrt{\lambda_t}$, where $\lambda_t = 3/2 S_{ij} S_{ij}/\sigma_S^2$. The system of equations (1.1)-(1.5) for elastic case ($\lambda_t = 0$) in expanded form is given in the authors' work in [17]. To solve the equations of the dynamics of an elastic-plastic medium according to the Godunov scheme, the Riemann problem solution in the elastic approximation is used. Let's denote $c^2 = (\partial p/\partial \rho)_s; f = 1/(\rho(\partial \varepsilon/\partial p)_\rho)$. After linearization of the energy equation, the linearized system of equations can be rewritten in the form of 11 transport equations [14],[17], using the Riemann invariants R_i, which remain constant on the corresponding characteristic c_i, where

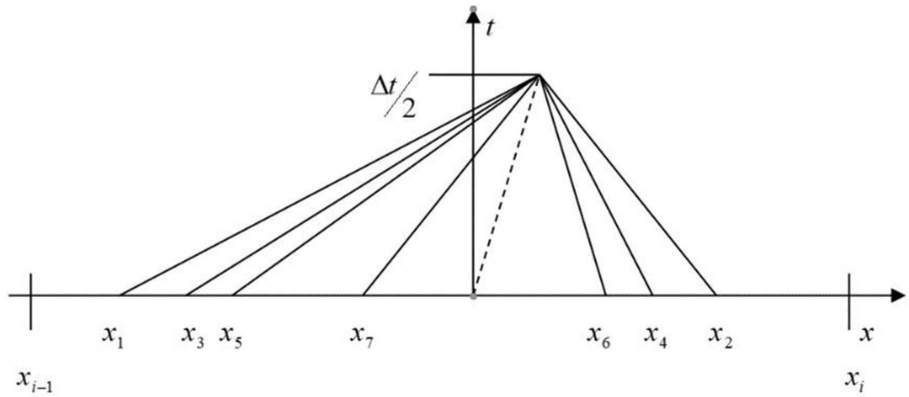
$$c_1 = u + a; c_2 = u - a; c_3 = u + \beta_y; c + 4 = u - \beta_y; c_5 = u + \beta_z; c_6 = u - \beta_z;$$

$$c_7 = c_8 = c_9 = c_{10} = c_{11} = u;$$

$$a^2 = c^2 + (4/3\mu - f S_{11})/\rho, \beta_2 = \sqrt{(\mu + 3/4 S_{11})/\rho - 0.5\sqrt{(0.25(S_{22} - S_{33})^2 + S_{23}^2)/\rho^2}}$$

$$\beta_3 = \sqrt{(\mu + 3/4 S_{11})/\rho + 0.5\sqrt{(0.25(S_{22} - S_{33})^2 + S_{23}^2)/\rho^2}}$$

The relationships on these characteristics are given in [14], [17]. For the first-order scheme, in the zone where the solution is sought (where the corresponding cell face is located), the corresponding invariants are determined and the parameters $\rho, u_1.u_2, u_3, p, S_{11}, S_{22}, S_{33}, S_{12}, S_{13}, S_{23}$ needed to calculate the flows are determined from them. To obtain a second-order approximation scheme, it is also necessary to interpolate the parameters, in this case the invariants. The parameters $U(\rho, u_1.u_2, u_3, p, S_{11}, S_{22}, S_{33}, S_{12}, S_{13}, S_{23})$ at the cell centers before calculating the invariants must be additionally corrected for the time layer $\Delta t/2$, taking into account the influence



$$x_1 = 0.5 * \Delta t * (w - u - a); x_3 = 0.5 * \Delta t * (w - u - \beta_y); x_5 = 0.5 * \Delta t * (w - u - \beta_z);$$

$$x_2 = 0.5 * \Delta t * (w - u + a); x_4 = 0.5 * \Delta t * (w - u + \beta_y); x_6 = 0.5 * \Delta t * (w - u + \beta_z);$$

$$x_7 = x_8 = x_9 = x_{10} = x_{11} = 0.5 * \Delta t * (w - u);$$

FIG 3. Coordinates of the interpolation points of the Riemann invariants for the second-order approximation scheme for an elastic system.

of the tangential (y and z) components of the equations, similar to the case for the Euler equations; the derivatives with respect to y and z in the right-hand sides for cells $i-1, j, k$ and i, j, k are calculated with the second-order of accuracy (over three points). For example, for the equation of motion of an elastic-plastic flow for the face $i-1/2, j, k$(the cell i, j, k is integrated), this correction will have the following form (the notations for the velocities are taken as in the formulas for the Euler equations) $u^*_{i-1} = u_{i-1} = \Delta t/2(v_0 u_y + w_0 u_z - S_{12,y}/\rho - S_{13,z}/\rho)$. Let's denote the invariants determined by these refined parameters U also by symbols with asterisks $R^*_n, n = 1, .., 11$. The coordinates of the interpolation points are defined as the boundaries of the regions of influence of the corresponding invariants on the position of the face at the moment of time $\Delta t/2$, where W the velocity of the face (indicated by the dotted line in Fig. 3), as follows:

$$x_n = 0.5(x_{i-1} + x_i) + (W - c_n)\Delta t/2, n = 1, .., 11 \tag{2.9}$$

Let's denote the points for interpolation as , and the invariants that are interpolated at these points with index "m", respectively:

$$R^m_n = R^{*,i-1}_n + (R^{*,i}_n - R^{*,i-1}_n)/(x_i - x_{i-1})(x_n - x_{i-1}), n = 1, .., 11$$

The flow parameters obtained from these invariants provide a second-order approximation of the scheme for the linearized equations. As a result, the approach providing the second-order of approximation of the linearized equations (1.1)-(1.5) in space and time consists of the following three steps (the boundary $i-1/2, j, k$ separating the cell $(i-1, j, k)$ and the cell (i, j, k). The first step is the determination of the interpolation points

x^+_{i-1} and x^+_i for the Euler equations and the points $x_n, n = 1,..,11$ for the elastic-plastic flow, respectively, according to formulas (2.7) and (2.9). The second step is the correction of the parameters $U(x_{i-1}, y_j, z_k)$ and $U(x_i, y_j, z_k)$ at the cell centers taking into account the tangential (y and z) gradients at time $\Delta t/2$ (definition of $U^*(x_{i-1}, y_j, z_k)$ and $U^*(x_i, y_j, z_k)$ and calculation of $R^{*,i-1}_n, R^{*,i}_n, n = 1,..,11$. The third step is the interpolation for the Euler equations of values $U^*(x_{i-1}, y_j, z_k)$ and $U^*(x_i, y_j, z_k)$ at points x^+_{i-1} and x^+_i and RP for $U^*(x^+_{i-1}, y_j, z_k)$ and $U^*(x^-_i, y_j, z_k)$ with the appropriate choice of $U(x_{i-1/2}, y_j, z_k)$. For elastic-plastic flows, this is the interpolation of $R^{*,i-1}_n, R^{*,i}_n$ into points $x_n, n = 1,..,11$ and the definition of $U(x_{i-1/2}, y_j, z_k)$ from invariants $R^m_n, n = 1,..,11$ and from the corresponding material constants and flow parameters interpolated to the point $x_7 = (x_{i-1}+x_i)/2+(W-c_7)\Delta t/2$ [17]. The corrector step of the modified scheme (flow calculation and integration) coincides with the first-order scheme for 3D equations. For elastic-plastic flows, the stress tensor correction is performed in accordance with [14].

3.4. Generalization to the nonlinear case, stability, implementation of boundary conditions with increased accuracy.

For generalization to the nonlinear case, the following approach is used. The linearization parameters should be chosen individually for each cell. For CFD we additionally consider the mass conservation equation with interpolation of densities similar to pressures. When determining the interpolation points, the influence of the gradients of sound and transport velocities, as well as the influence of the tangential gradients of the flow parameters, are taken into account, i.e., the coordinates of the interpolation points are determined after taking these gradients into account at $\Delta t/2$. For the stability of the modified scheme for 3D equations, the same criteria can be used as for the first-order 3D scheme. Analysis of the second-order 3D scheme for linearized equations shows that this modification is more stable than the first-order scheme. The stability criteria coincide with the Lax-Wendroff scheme. That is, for the 2D case on square grids the step will be in $\sqrt{2}$, and for 3D on cubic grids it will be $\sqrt{3}$ times larger than the step of the first-order scheme. Greater stability is achieved due to a more accurate approximation of the tangent components. Due to the compactness of the template, this modification also allows for an effective increase in accuracy when implementing various types of boundary conditions and contact interactions, in particular, when solving problems of interaction between gas-liquid media and elastic-plastic structures [18]. In this case, to solve RP at the boundary, the flow parameters for the Euler equations or the Riemann invariants for elastic-plastic bodies are extrapolated from the boundary cells to the corresponding points (boundaries of the influence regions).

3.5. Monotony, adjustable viscosity of the scheme.

In the area of discontinuous solutions, this modification has the disadvantages inherent in second-order approximation schemes - it is non-monotone. One of the advantages of this modification is the existence of an algorithm for eliminating the disadvantages in the area of discontinuous solutions by switching to the RP solution for the Godunov scheme of the first- order of accuracy. The problem is how to localize these discontinuous areas. One of the options for gas dynamics is to switch to the first-order RP in the compression region and this region is determined by comparing the pressure from the acoustic decay with the pressure in the lower time layer [10]. In this case, a significant viscosity is introduced into the scheme in the area of monotonic compression. Another method has also been tested - with a lower viscosity. In [12], a quadratic spline applied to pressure and density was used to determine a criterion for switching to the first-order RP. With a monotonic discrete solution, the spline constructed on this solution can already be non-monotonic, and the non-monotonicity of the spline is the criterion for switching to the RP of the first-order accuracy scheme. For example, when calculating the RP at an edge $x_{i-1/2}$, a left quadratic spline is constructed in pressures p_{i-2}, p_{i-1}, p_i and a right one in pressures p_{i-1}, p_i, p_{i+1}. If the left spline has an extremum, and this extremum is located between the centers of cells with the coordinates x_{i-2} and x_i, or the right spline has an extremum located between the centers of cells with the coordinates x_{i-1} and x_{i+1}, then the parameters from the cell centers are taken to solve RP. For elastic-plastic flows, in contrast to gas dynamics problems, where it is necessary to analyze pressure and density fields, to obtain monotonic solutions, it is sufficient to analyze the normal stress field [13]. From the analysis of the parametric differential approximation, it follows that it is possible to regulate the scheme viscosity of the resulting 3D scheme due to the modified 3D Riemann problem from a second-order scheme with zero viscosity to the viscosity of the Godunov scheme. By combining the viscosity terms due to the interpolation points and the viscosity from the tangential spatial corrections, one can obtain volume and shear viscosity terms for the linearized equations that are equal to the physical values.

4 TEST EXAMPLES

4.1. Interaction of two shock waves (Woodward test).

A collision of two strong shock waves is simulated [19]. The calculation was per-formed using uniform Eulerian grids. The computational domain is bounded by rigid walls, the initial parameters at $x < 0.1 p = 1000$, at $0.1 < x < 0.9 p = 0.01$, at $x > 0.9 p = 100$, the density, velocity and adiabatic index are the same for the entire domain: $\rho = 1, u = 0, \gamma = 1.4$.

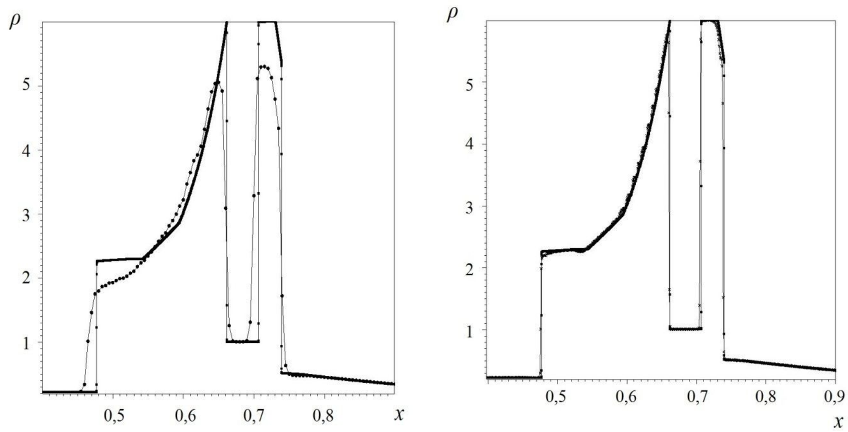


FIG 4. Density distributions, t=0.026, grids 200, 48000 and 1200, 48000 on the right.

The calculation was performed up to time t=0.038. At this time instant, a rather complex picture of the distribution of gas-dynamic parameters is formed, which is widely used to check the quality of schemes. Figures 4, 5, 6 show the density distributions along the x axis at times 0.026, 0.030 and 0.038, respectively, for grids 200 and 1200 (left) and 1200 and 48000 (right) ac-cording to the modified scheme. The results for 48000 cells are close to the solution [19]. The results for 1200 cells are marked with crosses and are close to the calculations for 48000 cells (squares). In this figure, each symbol (cross, square, circle) corresponds to one calculation cell. At the time 0.028, when the waves collide, the maximum pressure of 1020.9 and the maximum density of 28.5 are observed at the coordinate x=0.6941. The solution obtained by the modified scheme demonstrates good resolution in the region of contact discontinuities on the Eulerian grids. The region of smearing of contact discontinuities for the used grids is no more than three cells.

4.2. Forced elastic vibrations of a clamped plate.

A two-dimensional problem of deformation of an elastic plate OABC with a thickness of 5 cm and a length of 50 cm under the action of a suddenly applied load is considered (Fig. 7). The density of the material is 7.88 g/cm^3, the bulk modulus is 166 GPa, the shear modulus is 81.4 GPa. At t> 0, a constant pressure P = 1 MPa acts at the boundary AB, P = 0.1 MPa at the boundary BSO. The velocities at the boundary OA are $u = 0$ and $w = 0$, respectively, along the x and z axes (rigid fixation conditions). Figure 8 shows the results of calculations carried out using the Godunov scheme of the first and second-orders of accuracy with a safety factor for the

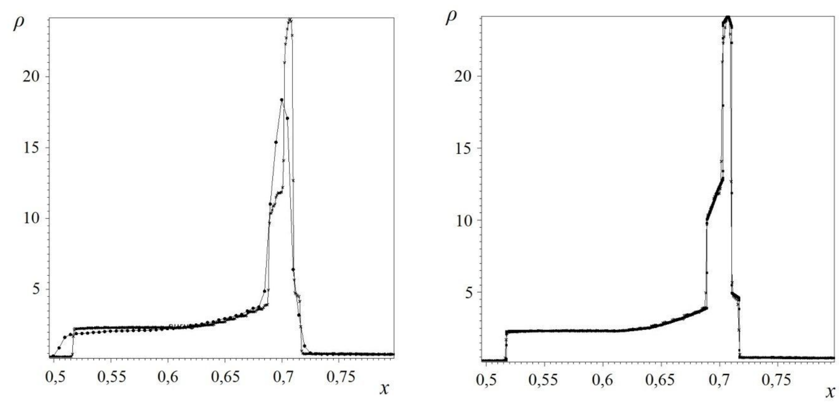


FIG 5. Density distributions, t=0.030, grids 200, 1200 and 1200, 48000 on the right.

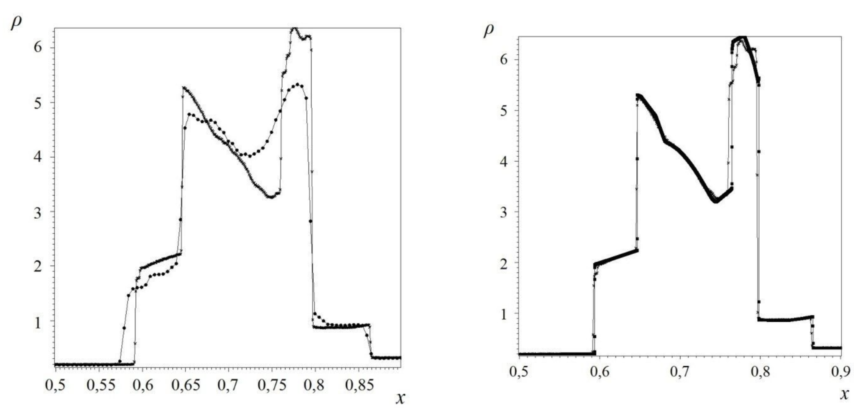


FIG 6. Density distributions, t=0.038, grids 200, 1200 and 1200, 48000 on the right.

integration step of K = 0.5. Numbers 1, 2, 3 mark the beam velocity time histories along the z axis at point B using the first-order scheme versus time for the 50x5, 100x10, and 200x20 computational grids, respectively, which required 50, 100, and 200 thousand computational steps. Numbers 4 and 5 mark the solution for 50x5 discretization using the second-order scheme. Number 4 marks the solution without refinement of parameters in the RP solution (invariants for RP are taken from the center of the boundary cell) at the boundary, Number 5 is the solution with refinement (the invariants are extrapolated from the boundary and preboundary cells). The first-order scheme yields a strongly damped solution. In the modified scheme, the refined implementation of the boundary conditions significantly improves the quality of the numerical solution; these results are virtually independent of the Courant number. Solution 5 virtually coincides with the solution using LS

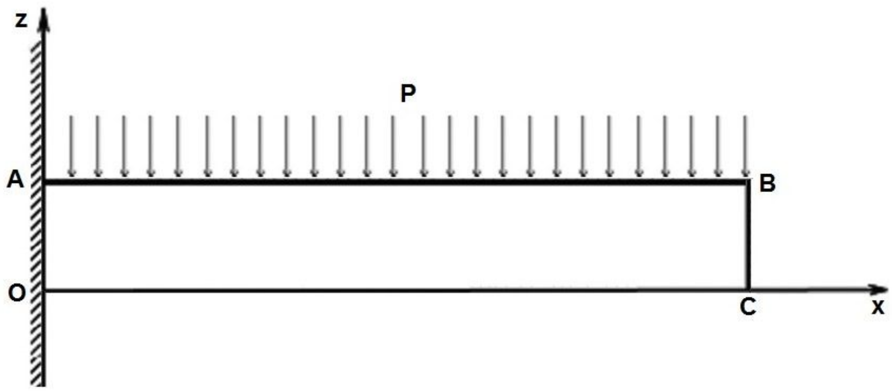


FIG 7. The problem statement.

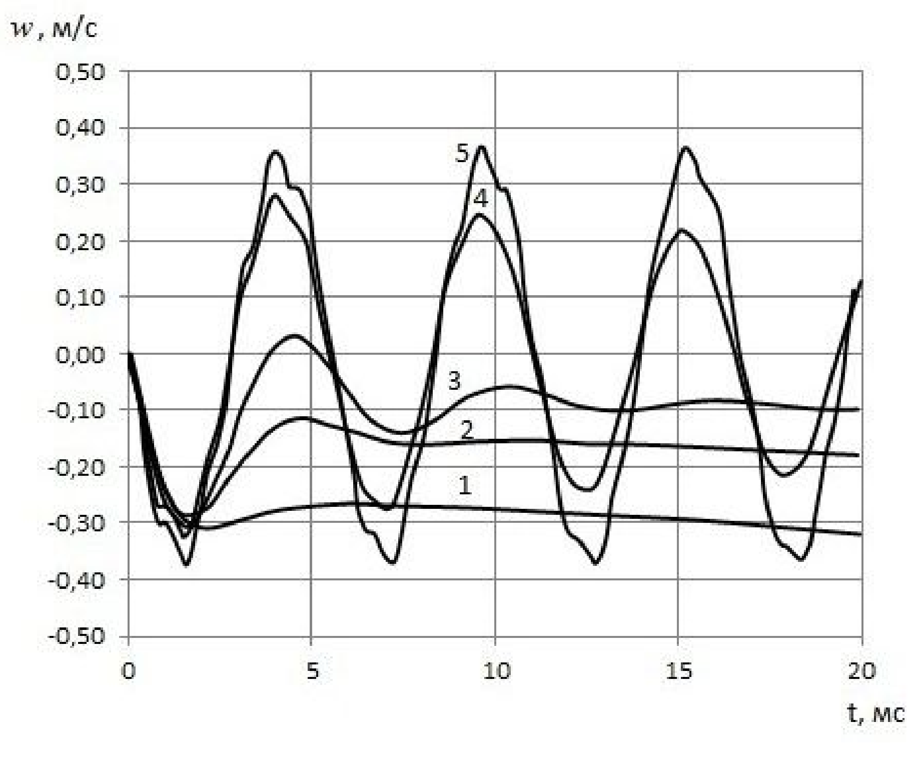


FIG 8. Vertical velocities time histories at point B.

DYNA in the ANSYS soft-ware suite. Based on the proposed methodology and multigrid algorithms, an effective software package has been developed for solving three-dimensional dynamic problems of interaction of gas-liquid media with elastic-plastic solids. Different problems of explosive and impact interaction of structural elements with compressible media were solved [14], [15], [17], [18], [20].

5 CONCLUSIONS

A parametric differential approximation for the linearized spatial Euler equations and the equations of dynamics of elastoplastic media is constructed as applied to the Godunov scheme. Based on the analysis of this differential approximation, a 3D difference scheme is proposed that has the second-order of accuracy in space and time on smooth solutions and monotonic behavior on discontinuities on a compact template for gas-dynamic and elasto-plastic flows in Eulerian variables. The accuracy is increased by modifying only the predictor step of the first-order scheme. The resulting modification of the scheme has an obvious phys-ical meaning - it brings together the areas of influence of the differential and difference problems. The main advantage of the modified scheme for CSD, which has the second-order of approximation in time and space for elastoplastic flows in Eulerian variables, is the use of only an elastic solution to the Riemann problem at the predictor stage, which significantly simplifies the solution of elastoplastic problems. The compact template of this modification simplifies the implementation of boundary conditions and allows increasing the accuracy at the boundary to the second-order of approximation, which is especially important for CSD. The implementation of this modification, unified for gas-liquid and elastic-plastic media, into existing software packages based on Godunov-type schemes, does not require significant software changes and allows solving different three-dimensional dynamic problems of interaction of gas-liquid media with elastic-plastic solids within the framework of one method.

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